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PLASTIC POTENTIAL THEORY IN LARGE STRAIN ELASTOPLASTICITY

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Abstract—The classical theory of isotropic, rate-independent, strain-hardening elastoplasticity based on the "invariance of elastic properties" is reformulated as an incremental state variable theory. In this Eulerian (spatial), formulation, state is described in terms of a scalar work-hardening parameter and a single, finite strain elastic deformation tensor that fixes the elastic stretch ellipsoid in the current configuration. This theoretically "efficient" formulation proves amenable to the consideration of material stability as the postulate of Il'iushin is shown to rigorously extend the classical plastic potential theory into the larger strain/deformation regime in virtually identical form. Special care is taken to establish the precise connection between these theoretical forms and those of Lee, as well as the more general forms of Green and Naghdi. Small strain linearization then serves to justify a particularly convenient frame-invariant generalization of the classical Prandtl–Reuss equations.

1. INTRODUCTION

In recent years, considerable effort has been expended in attempting to generalize the classical elastoplastic constitutive laws to account for large deformation and geometry change. In this paper, attention is focused on the simplest material model, due to Hill[1], in which plastic flow is characterized as isochoric rearrangement of material along slip planes. For this model, step removal of supporting stress is assumed to result in the instantaneous recovery of an isotropic "unstressed" element having elastic properties that are identical to those of a preselected reference element. It is also assumed that material response is rate independent with continuous deformation giving rise to continuous variation in energy and stress, and that continued plastic working results in increased resistance to further plastic flow owing to an isotropic increase in dislocation barriers.

The generation of a large deformation constitutive theory based on this specific set of material hypotheses was first attempted by Lee[2] and later by others including the present author[3]. The first difficulty, which was addressed in [3], is to generate a fully frame-invariant theory in which the significance of the particular choice of reference configuration is mathematically removed. This is required since the material hypotheses stipulate that an element's "state" in its current configuration be fixed by the elastic strain measured relative to an unstressed (stressfree) configuration and the accumulated plastic work. These previous considerations led to the conclusion that there must necessarily exist "a non-trivial coupling of the total and plastic strain tensors in the argument of a properly invariant energy function". In particular, it was demonstrated that within the context of the general theory of Green and Naghdi[4], the specification of an energy function having the convenient special form

$$\psi = \hat{\psi}(\mathbf{E}^*), \qquad \mathbf{E}^* \equiv \mathbf{E} - \mathbf{E}_p\dagger \tag{1}$$

does not ensure that "elastic properties are not coupled with plastic deformation", as claimed by Dafalias[5].‡ In fact, the type of dependence that is required to guarantee

† Dafalias[5] uses the notation $\mathbf{E}_e$ for the strain difference $\mathbf{E}^*$. This should not be confused with the "elastic strain", $\mathbf{E}_e = \frac{1}{2}(\mathbf{F}_e^T\mathbf{F}_e - \mathbf{I})$, identified by Naghdi and Trapp[6, eqn (32)].

‡ Various forms of this model have been suggested on a number of occasions, e.g. (1) Naghdi and Trapp[7, Sections 4 and 6], (2) Dafalias[5, Section 6], and (3) Casey and Naghdi[8, Section 4].

the invariance of elastic properties, in the sense intended by Hill[1], does not seem to be widely recognized as an important special case. This despite the fact that Naghdi and Trapp[6] provide a vehicle for such considerations through the introduction of the tensor **A** [see eqn (37)] in connection with a "special class" of materials. It is significant that this invariance property is easily incorporated into a state-variable formulation of the type considered in [9]. This is due to the fact that all general forms are expressed in terms of Eulerian (spatial as opposed to referential) stress and deformation measures with the body referred to its current configuration.

The second problem is to satisfactorily extend the plastic potential theory of Drucker[10]. Although this was partially resolved in [3], the Eulerian state-variable format adopted here leads to a significantly more efficient mathematical formulation that provides additional insight and allows for extension of the previous analysis. By first recasting this theory in this Eulerian state-variable form and then requiring stability in the sense of Il'iushin[11], it is now possible to rigorously prove (within the context of the aforementioned Hill model) the existence of an isotropic yield function depending on the Kirchhoff stress deviator and the accumulated plastic work,†

$$f = f(\acute{\boldsymbol{\tau}}, \omega_p), \qquad \acute{\boldsymbol{\tau}} = \operatorname{dev}(\boldsymbol{\tau}), \tag{2}$$

which determines the "direction" of a well-defined, deviatoric "plastic strain rate" through a plastic potential relationship of the form

$$\mathfrak{D} = \alpha \left(\frac{\partial f}{\partial \boldsymbol{\tau}}\right). \tag{3}$$

In Section 2 the general theoretical forms that are appropriate for the Hill model are set. These forms are taken from [9] and are based on the assumption that the "state" of a material element, in its present configuration, is determined by a single, symmetric, positive definite tensor **c** and a scalar hardening parameter ω_p. The deformation tensor **c** is taken to define the elastic stretch ellipsoid in the current configuration through the expression

$$\boldsymbol{\lambda}\cdot\mathbf{c}\cdot\boldsymbol{\lambda} = 1, \tag{4}$$

wherein any solution $\boldsymbol{\lambda} = \lambda\hat{\boldsymbol{\lambda}}$ determines the elastic stretch λ of the material fiber currently oriented in the $\hat{\boldsymbol{\lambda}}$ direction. This assumption not only represents the correct mathematical formulation of the material hypotheses, but also "builds in" significant theoretical economy through the elimination of one tensor deformation measure.

This section concludes with the specification of yield criteria in terms of a state-dependent yield function, the establishment of a number of crucial properties relating to the elastic "accessibility" of states, and the determination of the thermodynamic restrictions necessary to guarantee the dissipation inequality.

In Section 3, the connection between this "nonstandard" Eulerian formulation and other more traditional formulations is established. First, the relationships between the present Eulerian deformation measures and the Lagrangian finite strain measures associated with the Green–Naghdi format are determined. This makes it possible to recast this theory in its equivalent Lagrangian form and thereby demonstrate the relative economy of the present approach. An additional byproduct of this exercise is the conclusion that "invariance of elastic properties" implies an isotropic energy function of the form

$$\psi = \hat{\psi}(\mathbf{G}), \qquad \mathbf{G} \equiv \mathbf{C}^{-1/2}\mathbf{C}_p\mathbf{C}^{-1/2}, \tag{5}$$

† Recall that the Kirchhoff and Cauchy stress tensors are related through the expression $\boldsymbol{\tau} = (\rho_0/\rho)\boldsymbol{\sigma}$ in terms of the reference and current mass densities ρ_0 and ρ.

in terms of the total and plastic Green deformation tensors $\mathbf{C}$ and $\mathbf{C}_p$. Similarly, the relationships between the present deformation measures and the constituents of the $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ gradient decomposition make it possible to establish that the present theory and that proposed by Lee[2] are, in fact, equivalent.

The relationships established in Section 3 also make it possible to exploit the general Il'iushin stability results as presented by Hill and Rice[12] and more recently by Dafalias[5]. This is accomplished in Section 4 as the established constitutive inequalities are recast in terms of the present Eulerian measures. This leads directly to a "Drucker" inequality,

$$(\boldsymbol{\tau}_1 - \boldsymbol{\tau}_0)\cdot\mathcal{D} \geq 0, \tag{6}$$

and the attendant conclusions pertaining to convexity of yield surface and normality of plastic strain rate as embodied in eqn (3).

In the concluding section, an elastic log-strain measure is used to demonstrate that a hypoelastic rate equation of the form

$$\mathring{\boldsymbol{\tau}} = \underset{\sim}{\mathbf{H}}(\boldsymbol{\tau}, \omega_p)\cdot\mathbf{D} \tag{7}$$

approximates the exact theory to terms that are second order in elastic strain. It is noteworthy that this particular form, with slight modification of yield criteria, has been cited as being a convenient frame-invariant generalization of the classical Prandtl–Reuss equation inasmuch as it gives rise to a symmetric stiffness matrix in a finite element formulation.

2. GENERAL THEORY

As noted, the present material model is based on the assumption that an element's state in its current configuration is fixed by its instantaneous elastic distortion and the accumulated plastic work. In view of the description of elastic deformation and the general constitutive forms appearing in [9], it is possible to immediately write the forms

$$\psi = \hat{\psi}(\mathbf{c}) \tag{8a}$$

$$\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{c}) \tag{8b}$$

$$\mathring{\mathbf{c}} = \mathbf{c}(\mathbf{d} - \mathbf{D}) + (\mathbf{D} - \mathbf{d})\mathbf{c}, \qquad \mathbf{d} = \hat{\mathbf{d}}(\mathbf{c}, \omega_p, \mathbf{D}) \tag{8c}$$

$$\dot{\omega}_p = \nu(\mathbf{c}, \omega_p, \mathbf{D}) \tag{8d}$$

in terms of isotropic "response" and "evolution" functions $\hat{\psi}$, $\hat{\boldsymbol{\sigma}}$ and $\hat{\mathbf{d}}$, ν. In these expressions, ψ represents the elastic strain energy per unit mass, while $\boldsymbol{\sigma}$ represents Cauchy stress; $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$, the symmetric part of the velocity gradient tensor $\mathbf{L}$; $(\mathring{\ })$, the corotational or Jaumann derivative; and ω_p, the accumulated plastic work per unit mass defined through the mechanical dissipation inequality

$$\rho\dot{\omega}_p = \boldsymbol{\sigma}\cdot\mathbf{D} - \rho\dot{\psi} \geq 0. \tag{9}$$

The symmetric, positive definite deformation tensor $\mathbf{c}$ and the symmetric plastic deformation rate or "slippage" tensor

$$2\boldsymbol{\Gamma}_c = \mathbf{cd} + \mathbf{dc} \tag{10}$$

describe the local geometry and the rate of change of geometry, respectively, of the local unstressed configuration relative to the current. This is accomplished through the expressions

$$\mathbf{y}_1\cdot\mathbf{y}_2 = \mathbf{x}_1\cdot\mathbf{c}\cdot\mathbf{x}_2 \tag{11a}$$

$$\frac{d}{dt}(\mathbf{y}_1\cdot\mathbf{y}_2) = \mathbf{x}_1\cdot 2\mathbf{\Gamma}_c\cdot\mathbf{x}_2, \tag{11b}$$

which hold for any pair of flow-embedded ($\dot{\mathbf{x}} = \mathbf{L}\cdot\mathbf{x}$) material directors (fibers) $\mathbf{x}_1$ and $\mathbf{x}_2$ and their respective counterparts $\mathbf{y}_1$ and $\mathbf{y}_2$ in the unstressed configuration. In writing these forms, $\mathbf{D}$ and ω_p have been excluded as arguments of the response functions owing to the assumption that continuous deformation elicits continuous response and the assumed invariance of elastic properties with continued plastic flow. As shown in [9], the isochoric nature of the plastic deformation mechanism is modelled by imposing the additional constraints

$$\frac{\rho}{\rho_0} = [\det(\mathbf{c})]^{1/2} \tag{12a}$$

and

$$\mathbf{b}\cdot\mathbf{\Gamma}_c = \mathrm{tr}(\mathbf{d}) = 0, \qquad \mathbf{b} \equiv \mathbf{c}^{-1} \tag{12b}$$

in terms of the reference mass density and the inverse deformation tensor. To guarantee rate-independent material behavior, it is also necessary to require that the state evolution functions be homogeneous, degree one in the rate of deformation tensor, i.e.

$$\begin{aligned} \hat{\mathbf{d}}(\mathbf{c}, \omega_p, \kappa\mathbf{D}) &= \kappa\hat{\mathbf{d}}(\mathbf{c}, \omega_p, \mathbf{D}) \\ \nu(\mathbf{c}, \omega_p, \kappa\mathbf{D}) &= \kappa\nu(\mathbf{c}, \omega_p, \mathbf{D}). \end{aligned} \tag{13}$$

Before proceeding to introduce yield criteria, it shall prove convenient to rewrite these general forms in terms of new scalar and tensor variables and rate functions, viz.

$$\begin{aligned} \varphi &= \hat{\varphi}(\mathbf{c}) \equiv \rho_0\hat{\psi}(\mathbf{c}) \\ \boldsymbol{\tau} &= \hat{\boldsymbol{\tau}}(\mathbf{c}) \equiv \left(\frac{\rho_0}{\rho}\right)\hat{\boldsymbol{\sigma}}(\mathbf{c}) \\ \gamma &\equiv \rho_0\omega_p \\ \mathfrak{D} &= \hat{\mathfrak{D}}(\mathbf{c}, \gamma, \mathbf{D}) \equiv \mathbf{b}^{1/2}\,[\mathbf{c}\hat{\mathbf{d}}(\mathbf{c}, \gamma/\rho_0, \mathbf{D}) + \hat{\mathbf{d}}(\mathbf{c}, \gamma/\rho_0, \mathbf{D})\mathbf{c}]\mathbf{b}^{1/2} \\ \eta &= \hat{\eta}(\mathbf{c}, \gamma, \mathbf{D}) \equiv \rho_0\nu(\mathbf{c}, \gamma/\rho_0, \mathbf{D}), \end{aligned} \tag{14}$$

so that

$$\begin{aligned} 2\mathbf{\Gamma}_c &= \mathbf{c}\mathbf{d} + \mathbf{d}\mathbf{c} = \mathbf{c}^{1/2}2\mathfrak{D}\mathbf{c}^{1/2} \\ \dot{\gamma} &= \eta(\mathbf{c}, \gamma, \mathbf{D}).\dagger \end{aligned} \tag{15}$$

From eqn (11a) and the polar decomposition theorem, it is clear that stress relaxation, in the absence of material rotation, carries material directors $\mathbf{x}$ in the stressed con-

† Note that $\mathbf{d} = \mathfrak{D}$ whenever $\mathbf{c}$ and $\mathbf{d}$ commute. This property will be shown to result from subsequent stability considerations.

figuration into their unstressed counterparts

$$\mathbf{y} = \mathbf{c}^{1/2}\cdot\mathbf{x} \rightarrow \mathbf{x} = \mathbf{b}^{1/2}\cdot\mathbf{y}. \tag{16}$$

This correspondence makes it possible to rewrite the plastic deformation rate equation (11b) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{y}_1\cdot\mathbf{y}_2) = \mathbf{y}_1\cdot 2\mathfrak{D}\cdot\mathbf{y}_2. \tag{17}$$

Comparison of this with the rate of deformation equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}_1\cdot\mathbf{x}_2) = \mathbf{x}_1\cdot 2\mathbf{D}\cdot\mathbf{x}_2,$$

unambiguously identifies $\mathfrak{D}$ as the rate of deformation tensor for the particular unstressed configuration obtained through rotationfree stress relaxation as determined by eqn (16). In view of the definitions (14), it is also clear that the incompressibility constraint (12b) takes the form

$$\mathrm{tr}(\mathfrak{D}) = 0, \tag{18}$$

and that the mechanical dissipation (plastic working per unit unstressed volume) inequality (9) is given by

$$\dot{\gamma} = \eta = \boldsymbol{\tau}\cdot\mathbf{D} - \dot{\varphi} \geq 0 \tag{19}$$

in terms of the Kirchhoff stress $\boldsymbol{\tau}$.

2.1. *Yield criteria*

To incorporate the familiar concept of yield, it is assumed that there exists a smooth, scalar function (necessarily an isotropic function)

$$g = g(\mathbf{c}, \gamma), \tag{20}$$

which defines, for each γ, a closed connected *elastic region*,

$$C(\gamma) = \{\mathbf{c}: g(\mathbf{c}, \gamma) \leq 0\}, \tag{21}$$

in $\mathbf{c}$-space with boundary

$$\partial C(\gamma) = \{\mathbf{c}: g(\mathbf{c}, \gamma) = 0\} \tag{22}$$

defining states of elastic-plastic transition. Any deformation process

$$\mathbf{F} = \mathbf{F}(t), \qquad t_1 \leq t \leq t_2,$$

such that

$$\mathbf{F}(t_1) = \mathbf{I}$$

initiating at an elastic state $(\mathbf{c}_1, \gamma_1)$,

$$\mathbf{c}_1 \in C(\gamma_1) \rightarrow g(\mathbf{c}_1, \gamma_1) \leq 0,$$

is now assumed to be purely elastic, i.e.

$$\begin{aligned} \mathbf{c}(t) &= \mathbf{c}_e(t) \equiv [\mathbf{F}^{-1}(t)]^T \mathbf{c}_1 [\mathbf{F}^{-1}(t)] \\ \gamma(t) &= \gamma_1, \end{aligned} \tag{23}$$

if and only if

$$\{\mathbf{c}_e(t): t_1 \le t \le t_2\} \subseteq C(\gamma_1) \rightarrow g[\mathbf{c}_e(t), \gamma_1] \le 0, \qquad \text{for each } t: t_1 \le t \le t_2. \tag{24}$$

It therefore follows that a smooth deformation proceeds elastically through $(\mathbf{c}, \gamma)$, i.e.

$$[\mathfrak{D}, \eta]_{(c,\gamma,D)} = 0,$$

if and only if

$$g(\mathbf{c}, \gamma) < 0$$

or

$$g(\mathbf{c}, \gamma) = 0 \tag{25}$$

and

$$\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)_{\text{elastic}} = \left(\frac{\partial g}{\partial \mathbf{c}}\right) \cdot [\mathring{\mathbf{c}}]_{\mathfrak{D}=0} = -2\left[\frac{\partial g}{\partial \mathbf{c}}\mathbf{c}\right] \cdot \mathbf{D} \le 0.$$

Conversely, it must also follow that a smooth deformation proceeds "inelastically" through $(\mathbf{c}, \gamma)$, i.e.

$$[\mathfrak{D}, \eta]_{(c,\gamma,D)} \neq 0,$$

if and only if

$$g(\mathbf{c}, \gamma) > 0$$

or (26)

$$g(\mathbf{c}, \gamma) = 0 \quad \text{and} \quad \left(\frac{\partial g}{\partial \mathbf{c}}\mathbf{c}\right) \cdot \mathbf{D} < 0.$$

To guarantee that it will be possible to proceed elastically from any attainable state and to sustain a program of plastic loading through any transition state $\mathbf{c} \in \partial C(\gamma)$, it is necessary and sufficient to exclude the first of these possibilities by requiring that $(\partial g/\partial \mathbf{c})\, \mathbf{c}$ be nonvanishing at transition states,

$$g(\mathbf{c}, \gamma) = 0 \rightarrow \frac{\partial g}{\partial \mathbf{c}}\mathbf{c} \neq 0,$$

and that

$$\dot{g} = \frac{\partial g}{\partial \mathbf{c}} \cdot \mathring{\mathbf{c}} + \frac{\partial g}{\partial \gamma}\eta = 0 \tag{27}$$

during plastic loading, i.e. whenever

$$g(\mathbf{c}, \gamma) = 0 \quad \text{and} \quad \left(\frac{\partial g}{\partial \mathbf{c}}\mathbf{c}\right) \cdot \mathbf{D} < 0.$$

It shall prove useful to establish two topological properties relating to the accessibility of states in the elastic region $C(\gamma)$. The first property, namely, that

$$\mathbf{c} \in C(\gamma)$$

if and only if (28)

$$\bar{\mathbf{c}} = \mathbf{QcQ}^T \in C(\gamma)$$

for any proper orthogonal $\mathbf{Q}$, follows immediately from the isotropy of the yield function g. The second property guarantees that any pair of c-values $\mathbf{c}_1$, $\mathbf{c}_2 \in C(\gamma)$ which are accessible through a particular elastic deformation will, in fact, be accessible through a multiplicity of such paths. Specifically, for any pair of c-values $\mathbf{c}_1$, $\mathbf{c}_2 \in C(\gamma)$ in a *connected* elastic region $C(\gamma)$ and any proper orthogonal $\mathbf{Q}$, there exists an elastic deformation

$$\mathbf{F} = \mathbf{F}(t), \qquad t_1 \le t \le t_2,$$

with

$$\mathbf{F}(t_1) = \mathbf{I}$$
$$\mathbf{F}(t_2) = \mathbf{b}_2^{1/2}\mathbf{Q}\mathbf{c}_1^{1/2},$$

leading from state $(\mathbf{c}_1, \gamma)$ to $(\mathbf{c}_2, \gamma)$. This result follows from the connectedness of $C(\gamma)$, which establishes the existence of an elastic c-trajectory

$$\mathbf{c} = \mathbf{c}(t), \qquad t_1 \le t \le t_2,$$

with

$$\mathbf{c}(t_1) = \mathbf{c}_1 \quad \text{and} \quad \mathbf{c}(t_2) = \mathbf{c}_2,$$

and the fact that the imposed deformation

$$\mathbf{F} = \mathbf{F}(t) = \mathbf{b}^{1/2}(t)\mathbf{Q}(t)\mathbf{c}_1^{1/2}, \qquad t_1 \le t \le t_2, \tag{29}$$

via (23) and (24), tracks this same elastic trajectory, provided only that $\mathbf{Q} = \mathbf{Q}(t)$, $(t_1 \le t \le t_2)$ is proper orthogonal and satisfies the initial condition $\mathbf{Q}(t_1) = \mathbf{I}$.

2.2. *Thermodynamic restrictions*

The evolution equations (8c,d) with (15) can now be used to expand the time rates of the energy and yield functions:

$$\begin{aligned}
\dot{\varphi} &= \frac{\partial \varphi}{\partial \mathbf{c}} \cdot \mathring{\mathbf{c}} \\
&= \frac{\partial \varphi}{\partial \mathbf{c}} \cdot [\mathbf{c}^{1/2} 2\mathfrak{D} \mathbf{c}^{1/2} - \mathbf{cD} - \mathbf{Dc}] \\
&= 2\left[\mathbf{c}^{1/2}\frac{\partial \varphi}{\partial \mathbf{c}}\mathbf{c}^{1/2}\right] \cdot \mathfrak{D} - \left[\frac{\partial \varphi}{\partial \mathbf{c}}\mathbf{c} + \mathbf{c}\frac{\partial \varphi}{\partial \mathbf{c}}\right] \cdot \mathbf{D} \\
&= -2\left(\frac{\partial \varphi}{\partial \mathbf{c}}\mathbf{c}\right) \cdot (\mathbf{D} - \mathfrak{D}),
\end{aligned} \tag{30}$$

and similarly,

$$\dot{g} = -2\left(\frac{\partial g}{\partial \mathbf{c}}\mathbf{c}\right)\cdot(\mathbf{D} - \mathfrak{D}) + \frac{\partial g}{\partial \gamma}\eta.$$

The energy expansion can then be used to rewrite the dissipation inequality (19) in the form

$$\dot{\gamma} = \boldsymbol{\tau}\cdot\mathbf{D} - \dot{\varphi} = \left[\boldsymbol{\tau} + 2\left(\frac{\partial \varphi}{\partial \mathbf{c}}\mathbf{c}\right)\right]\cdot\mathbf{D} - 2\left(\frac{\partial \varphi}{\partial \mathbf{c}}\mathbf{c}\right)\cdot\mathfrak{D} \geq 0.$$

This leads to the conclusion that

$$\boldsymbol{\tau} = -2\left(\frac{\partial \varphi}{\partial \mathbf{c}}\mathbf{c}\right), \tag{31a}$$

and

$$\dot{\gamma} = \eta = \boldsymbol{\tau}\cdot\mathfrak{D} \geq 0, \tag{31b}$$

by virtue of the continuous dependence of $\boldsymbol{\tau}$ on $\mathbf{c}$ and the fact that each attainable state $(\mathbf{c}, \gamma)$ is a limit point of the open interior of the elastic region $C(\gamma)$.

After collecting results, and introducing the "yield gradient"

$$\boldsymbol{\tau}_g \equiv -2\left(\frac{\partial g}{\partial \mathbf{c}}\mathbf{c}\right), \tag{32}$$

the general theory embodied in (8) is now seen to take the form

$$\varphi = \hat{\varphi}(\mathbf{c}), \qquad \rho = \rho_0[\det(\mathbf{c})]^{1/2} \tag{33a}$$

$$\boldsymbol{\tau} = -2\left(\frac{\partial \varphi}{\partial \mathbf{c}}\mathbf{c}\right) \tag{33b}$$

$$\overset{\circ}{\mathbf{c}} = -(\mathbf{c}\mathbf{D} + \mathbf{D}\mathbf{c}) + 2\boldsymbol{\Gamma}_c, \qquad \boldsymbol{\Gamma}_c = \mathbf{c}^{1/2}\hat{\mathfrak{D}}(\mathbf{c}, \gamma, \mathbf{D})\mathbf{c}^{1/2}, \qquad \mathrm{tr}(\mathfrak{D}) = 0 \tag{33c}$$

$$\dot{\gamma} = \boldsymbol{\tau}\cdot\mathfrak{D} \geq 0, \tag{33d}$$

with

$$\mathfrak{D} = 0, \text{ whenever } \begin{cases} g(\mathbf{c}, \gamma) < 0 \text{ or} \\ g(\mathbf{c}, \gamma) = 0 \text{ and } \boldsymbol{\tau}_g\cdot\mathbf{D} \leq 0, \end{cases} \tag{34}$$

and

$$\mathfrak{D} \neq 0 \text{ and } \dot{g} = \boldsymbol{\tau}_g\cdot(\mathbf{D} - \mathfrak{D}) + \frac{\partial g}{\partial \gamma}(\boldsymbol{\tau}\cdot\mathfrak{D}) = 0,$$
$$\text{whenever } g(\mathbf{c}, \gamma) = 0 \text{ and } \boldsymbol{\tau}_g\cdot\mathbf{D} > 0.$$

Thus, a particular model is fixed with the specification of three isotropic functions: a scalar energy function $\hat{\varphi}(\mathbf{c})$, a scalar yield function $g(\mathbf{c}, \gamma)$ and a symmetric, deviatoric tensor "plastic deformation rate" function $\hat{\mathfrak{D}}(\mathbf{c}, \gamma, \mathbf{D})$. The choice of these functions must be consistent with all of the above equations and inequalities, with $\mathbf{c} = \mathbf{I}$ defining the class of stressfree states that produce a global minimum for internal energy.

It is now possible to demonstrate how the requirement of Il'iushin stability[11] severely restricts the form of the yield function and, in fact, fixes the plastic deformation rate $\mathfrak{D}$ in terms of $\hat{\varphi}$ and g. The development is exactly analogous to the small strain development of Drucker[10], and gives rise to generalized plastic potential theory for large deformation elastoplasticity. These results are formally identical to those obtained by Drucker, except that Cauchy stress $\boldsymbol{\sigma}$ and the plastic strain rate $\dot{\boldsymbol{\epsilon}}_p$ are replaced, respectively, by Kirchhoff stress $\boldsymbol{\tau}$ and $\mathfrak{D}$, and that Il'iushin stability does not preclude the possibility of a strain-softening material.

3. EQUIVALENT FORMS

Since these Eulerian or "rheological" constitutive relations are nonstandard in the field of solid mechanics, it may prove instructive to recast them in various equivalent (and more familiar) forms. This will facilitate comparison with existing theories and make it possible to exploit the general stability results of Hill and Rice[12] pertaining to Il'iushin stability.

The standard kinematic formulation of elastoplasticity is based on the notion of reference, plastically deformed and current material element configurations. The orientation and geometry of each of these configurations are described by the plastic, elastic and total deformation gradients $\mathbf{F}_p$, $\mathbf{F}_e$ and $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ and the associated deformation and rotation measures

$$\begin{aligned} \mathbf{Y} &= \mathbf{F}_p\cdot\mathbf{X},\ \mathbf{C}_p = \mathbf{F}_p^T\mathbf{F}_p,\ \mathbf{B}_p = \mathbf{C}_p^{-1},\ \mathbf{U}_p = \mathbf{C}_p^{1/2},\ \mathbf{F}_p = \mathbf{R}_p\mathbf{U}_p \\ \mathbf{x} &= \mathbf{F}_e\cdot\mathbf{Y},\ \mathbf{C}_e = \mathbf{F}_e^T\mathbf{F}_e,\ \mathbf{B}_e = \mathbf{C}_e^{-1},\ \mathbf{U}_e = \mathbf{C}_e^{1/2},\ \mathbf{F}_e = \mathbf{R}_e\mathbf{U}_e \\ \mathbf{x} &= \mathbf{F}\cdot\mathbf{X},\quad \mathbf{C} = \mathbf{F}^T\mathbf{F},\quad \mathbf{B} = \mathbf{C}^{-1},\quad \mathbf{U} = \mathbf{C}^{1/2},\quad \mathbf{F} = \mathbf{R}\mathbf{U}, \end{aligned} \tag{35}$$

in terms of the reference, plastically deformed and current material director representations $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{x}$. To establish the relationship between these configurations and the unstretched configuration defined through eqn (16), it is necessary to require that the unstretched and plasticity deformed directors $\mathbf{y}$ and $\mathbf{Y}$ be related through an orthogonal transformation, i.e.

$$\mathbf{y} = \mathbf{Q}\cdot\mathbf{Y},\qquad \mathbf{Y} = \mathbf{Q}^T\cdot\mathbf{y}. \tag{36}$$

Having thus imposed an identical geometry on the plastically deformed and unstretched configurations, it is then possible to determine an alternative gradient decomposition

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p = \overline{\mathbf{F}}_e\overline{\mathbf{F}}_p \tag{37}$$

in terms of

$$\begin{aligned} \mathbf{y} &= \overline{\mathbf{F}}_p\cdot\mathbf{X},\quad \overline{\mathbf{C}}_p = \overline{\mathbf{F}}_p^T\overline{\mathbf{F}}_p,\quad \overline{\mathbf{B}}_p = \overline{\mathbf{C}}_p^{-1},\quad \overline{\mathbf{U}}_p = \overline{\mathbf{C}}_p^{1/2},\quad \overline{\mathbf{F}}_p = \overline{\mathbf{R}}_p\overline{\mathbf{U}}_p \\ \mathbf{x} &= \overline{\mathbf{F}}_e\cdot\mathbf{y},\quad \overline{\mathbf{C}}_e = \overline{\mathbf{F}}_e^T\overline{\mathbf{F}}_e,\quad \overline{\mathbf{B}}_e = \overline{\mathbf{C}}_e^{-1},\quad \overline{\mathbf{U}}_e = \overline{\mathbf{C}}_e^{1/2},\quad \overline{\mathbf{F}}_e = \overline{\mathbf{R}}_e\overline{\mathbf{U}}_e. \end{aligned} \tag{38}$$

Now, since

$$\begin{aligned} \overline{\mathbf{F}}_e &= \mathbf{F}_e\mathbf{Q}^T \\ &= (\mathbf{R}_e\mathbf{U}_e\mathbf{R}_e^T)(\mathbf{R}_e\mathbf{Q}^T), \end{aligned}$$

with $\overline{\mathbf{F}}_e = \mathbf{b}^{1/2} = \overline{\mathbf{F}}_e^T$, by virtue of eqn (16), it follows from the uniqueness of the polar decomposition that

$$\begin{aligned} \mathbf{Q} = \mathbf{R}_e &\rightarrow \mathbf{y} = \mathbf{R}_e\cdot\mathbf{Y},\qquad \mathbf{Y} = \mathbf{R}_e^T\cdot\mathbf{y} \\ \mathbf{b}^{1/2} &= \overline{\mathbf{F}}_e = \mathbf{F}_e\mathbf{R}_e^T = \mathbf{R}_e\mathbf{U}_e\mathbf{R}_e^T \end{aligned}$$

and

$$\begin{aligned}\bar{\mathbf{F}}_p &= \bar{\mathbf{F}}_e^{-1}\mathbf{F}\\ &= \mathbf{R}_e\mathbf{F}_e^{-1}(\mathbf{F}_e\mathbf{F}_p)\\ &= (\mathbf{R}_e\mathbf{R}_p)\mathbf{U}_p.\end{aligned}$$

Thus,

$$\begin{aligned}
\bar{\mathbf{F}}_e &= \mathbf{F}_e\mathbf{R}_e^T = \mathbf{b}^{1/2} & \bar{\mathbf{F}}_p &= \mathbf{R}_e\mathbf{F}_p\\
\bar{\mathbf{C}}_e &= \mathbf{R}_e\mathbf{C}_e\mathbf{R}_e^T = \mathbf{b} & \bar{\mathbf{C}}_p &= \mathbf{C}_p\\
\bar{\mathbf{B}}_e &= \mathbf{R}_e\mathbf{B}_e\mathbf{R}_e^T = \mathbf{c} & \bar{\mathbf{B}}_p &= \mathbf{B}_p\\
\bar{\mathbf{U}}_e &= \mathbf{R}_e\mathbf{U}_e\mathbf{R}_e^T = \mathbf{b}^{1/2} & \bar{\mathbf{U}}_p &= \mathbf{U}_p\\
\bar{\mathbf{R}}_e &= \mathbf{I} & \bar{\mathbf{R}}_p &= \mathbf{R}_e\mathbf{R}_p.
\end{aligned} \tag{39}$$

Another useful correspondence follows from the relationship

$$\begin{aligned}\mathbf{y} &= \mathbf{R}_e\cdot\mathbf{Y}\\ \mathbf{c}^{1/2}\cdot\mathbf{x} &= \mathbf{R}_e\mathbf{F}_p\cdot\mathbf{X}\\ \mathbf{c}^{1/2}\mathbf{F}\cdot\mathbf{X} &= \mathbf{R}_e\mathbf{F}_p\cdot\mathbf{X},\end{aligned}$$

which implies that

$$\begin{aligned}\bar{\mathbf{F}}_p &= \mathbf{R}_e\mathbf{F}_p = \mathbf{c}^{1/2}\mathbf{F}\\ \bar{\mathbf{C}}_p &= \mathbf{C}_p = \mathbf{F}^T\mathbf{c}\mathbf{F},\end{aligned}$$

and

$$\begin{aligned}\mathbf{c} &= (\mathbf{F}^{-1})^T\mathbf{C}_p(\mathbf{F}^{-1})\\ &= \mathbf{R}\mathbf{U}^{-1}\mathbf{C}_p\mathbf{U}^{-1}\mathbf{R}^T\\ \mathbf{c} &= \mathbf{R}\mathbf{G}\mathbf{R}^T, \qquad \mathbf{b} = \mathbf{R}\mathbf{G}^{-1}\mathbf{R}^T,\end{aligned} \tag{40}$$

in terms of the new Lagrangian deformation measure

$$\mathbf{G} \equiv \mathbf{U}^{-1}\mathbf{C}_p\mathbf{U}^{-1}. \tag{41}$$

Moreover, since

$$\mathbf{c}^{1/2} = \mathbf{R}\mathbf{G}^{1/2}\mathbf{R}^T, \qquad \mathbf{b}^{1/2} = \mathbf{R}\mathbf{G}^{-1/2}\mathbf{R}^T,$$

it also follows that

$$\bar{\mathbf{F}}_e = \mathbf{F}_e\mathbf{R}_e^T = \mathbf{b}^{1/2} = \mathbf{R}\mathbf{G}^{-1/2}\mathbf{R}^T \tag{42a}$$

$$\bar{\mathbf{F}}_p = \mathbf{R}_e\mathbf{F}_p = \mathbf{c}^{1/2}\mathbf{F} = \mathbf{R}\mathbf{G}^{1/2}\mathbf{U}. \tag{42b}$$

A number of deformation rates also appear in the literature. In terms of the Green strain tensors

$$
\begin{aligned}
2\mathbf{E} &= \mathbf{C} - \mathbf{I} \\
2\mathbf{E}_p &= \mathbf{C}_p - \mathbf{I},
\end{aligned}
\tag{43}
$$

it is easy to verify that material deformation in the current and plastically deformed configurations is described by

$$
\begin{aligned}
&\frac{d}{dt}(\mathbf{x}_1\cdot\mathbf{x}_2) = \mathbf{X}_1\cdot 2\dot{\mathbf{E}}\cdot\mathbf{X}_2 \\
&\frac{d}{dt}(\mathbf{Y}_1\cdot\mathbf{Y}_2) = \frac{d}{dt}(\mathbf{y}_1\cdot\mathbf{y}_2) = \mathbf{X}_1\cdot 2\dot{\mathbf{E}}_p\cdot\mathbf{X}_2.
\end{aligned}
\tag{44}
$$

In terms of the "velocity gradients"

$$
\begin{aligned}
\mathbf{L} &= \dot{\mathbf{F}}\mathbf{F}^{-1} \\
\mathbf{L}_p &= \dot{\mathbf{F}}_p\mathbf{F}_p^{-1} \\
\overline{\mathbf{L}}_p &= \dot{\overline{\mathbf{F}}}_p\overline{\mathbf{F}}_p^{-1}
\end{aligned}
\tag{45}
$$

and the corresponding symmetric $(\ )_S$ and antisymmetric $(\ )_A$ parts

$$
\begin{aligned}
\mathbf{D} &= \mathbf{L}_S, & \mathbf{D}_p &= (\mathbf{L}_p)_S, & \overline{\mathbf{D}}_p &= (\overline{\mathbf{L}}_p)_S \\
\mathbf{W} &= \mathbf{L}_A, & \mathbf{W}_p &= (\mathbf{L}_p)_A, & \overline{\mathbf{W}}_p &= (\overline{\mathbf{L}}_p)_A,
\end{aligned}
\tag{46}
$$

it follows that these same material deformations are also described by

$$
\begin{aligned}
&\dot{\mathbf{x}} = \mathbf{L}\cdot\mathbf{x} \rightarrow \frac{d}{dt}(\mathbf{x}_1\cdot\mathbf{x}_2) = \mathbf{x}_1\cdot 2\mathbf{D}\cdot\mathbf{x}_2 \\
&\dot{\mathbf{Y}} = \mathbf{L}_p\cdot\mathbf{Y} \rightarrow \frac{d}{dt}(\mathbf{Y}_1\cdot\mathbf{Y}_2) = \mathbf{Y}_1\cdot 2\mathbf{D}_p\cdot\mathbf{Y}_2 \\
&\dot{\mathbf{y}} = \overline{\mathbf{L}}_p\cdot\mathbf{y} \rightarrow \frac{d}{dt}(\mathbf{y}_1\cdot\mathbf{y}_2) = \mathbf{y}_1\cdot 2\overline{\mathbf{D}}_p\cdot\mathbf{y}_2.
\end{aligned}
\tag{47}
$$

Comparison of these rate expressions with (44) and (17), in light of the established director relationships, leads to the conclusion that

$$
\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{D}\mathbf{F} \tag{48a}
$$

$$
\dot{\mathbf{E}}_p = \mathbf{F}_p^T\mathbf{D}_p\mathbf{F}_p = \overline{\mathbf{F}}_p^T\mathfrak{D}\overline{\mathbf{F}}_p, \tag{48b}
$$

and

$$
\mathbf{D} = (\mathbf{F}^{-1})^T\dot{\mathbf{E}}(\mathbf{F}^{-1}) = \mathbf{R}(\mathbf{U}^{-1}\dot{\mathbf{E}}\mathbf{U}^{-1})\mathbf{R}^T \tag{48c}
$$

$$
\mathfrak{D} = \mathbf{R}_e\mathbf{D}_p\mathbf{R}_e^T = (\overline{\mathbf{F}}_p^{-1})^T\dot{\mathbf{E}}_p(\overline{\mathbf{F}}_p^{-1}) = \overline{\mathbf{R}}_p(\mathbf{U}_p^{-1}\dot{\mathbf{E}}_p\mathbf{U}_p^{-1})\overline{\mathbf{R}}_p^T. \tag{48d}
$$

Observe that the plastic deformation rate $\overline{\mathbf{D}}_p$, which plays a prominent role in the Lee formulation (see [2]), has been replaced through the identification

$$
\overline{\mathbf{D}}_p = \mathfrak{D}. \tag{49}
$$

Also be apprised that the deformation and deformation rate expressions [(39), (42), (48)], in terms of $\mathbf{G} = \mathbf{U}^{-1}\mathbf{C}_p\mathbf{U}^{-1} = \mathbf{R}^T\mathbf{c}\mathbf{R}$, and the defining expressions [(35), (38), (43), (45)] shall henceforth be used without citing reference.

3.1. *The Green–Naghdi connection*

In light of the Eulerian constitutive forms [(33), (34)] and the established deformation and deformation rate relationships, it follows that

$$\varphi = \hat{\varphi}(\mathbf{E}, \mathbf{E}_p) \equiv \hat{\varphi}(\mathbf{G}) = \hat{\varphi}(\mathbf{c}) \tag{50a}$$

$$g = \hat{g}(\mathbf{E}, \mathbf{E}_p, \gamma) \equiv g(\mathbf{G}, \gamma) = g(\mathbf{c}, \gamma) \tag{50b}$$

$$\mathbf{T} \equiv -2\left(\frac{\partial\varphi}{\partial\mathbf{G}}\mathbf{G}\right) = \mathbf{R}^T\left[-2\left(\frac{\partial\varphi}{\partial\mathbf{c}}\mathbf{c}\right)\right]\mathbf{R} = \mathbf{R}^T\boldsymbol{\tau}\mathbf{R} \tag{50c}$$

$$\mathbf{T}_g \equiv -2\left(\frac{\partial g}{\partial\mathbf{G}}\mathbf{G}\right) = \mathbf{R}^T\left[-2\left(\frac{\partial g}{\partial\mathbf{c}}\mathbf{c}\right)\right]\mathbf{R} = \mathbf{R}^T\boldsymbol{\tau}_g\mathbf{R} \tag{50d}$$

$$\begin{aligned}\boldsymbol{\tau}_g\cdot\mathbf{D} &= \boldsymbol{\tau}_g\cdot\mathbf{R}\mathbf{U}^{-1}\dot{\mathbf{E}}\mathbf{U}^{-1}\mathbf{R}^T\\ &= (\mathbf{R}^T\boldsymbol{\tau}_g\mathbf{R})\cdot(\mathbf{U}^{-1}\dot{\mathbf{E}}\mathbf{U}^{-1}) = \mathbf{T}_g\cdot(\mathbf{U}^{-1}\dot{\mathbf{E}}\mathbf{U}^{-1}),\end{aligned} \tag{50e}$$

and

$$\begin{aligned}\dot{\mathbf{E}}_p &= \overline{\mathbf{F}}_p^T\hat{\mathfrak{D}}(\mathbf{c}, \gamma, \mathbf{D})\overline{\mathbf{F}}_p\\ &= \mathbf{U}\mathbf{G}^{1/2}\mathbf{R}^T\hat{\mathfrak{D}}(\mathbf{c}, \gamma, \mathbf{D})\mathbf{R}\mathbf{G}^{1/2}\mathbf{U}\\ &= \mathbf{U}\mathbf{G}^{1/2}\hat{\mathfrak{D}}(\mathbf{R}^T\mathbf{c}\mathbf{R}, \gamma, \mathbf{R}^T\mathbf{D}\mathbf{R})\mathbf{G}^{1/2}\mathbf{U}\\ \dot{\mathbf{E}}_p &= \mathbf{U}\mathbf{G}^{1/2}\hat{\mathfrak{D}}(\mathbf{G}, \gamma, \mathbf{U}^{-1}\dot{\mathbf{E}}\mathbf{U}^{-1})\mathbf{G}^{1/2}\mathbf{U}\end{aligned} \tag{51}$$

by virtue of the isotropy of the scalar and tensor functions $\hat{\varphi}$, g and $\hat{\mathfrak{D}}$. Notice that the new Lagrangian stress variables $\mathbf{T}$ and $\mathbf{T}_g$ introduced above commute with $\mathbf{G}$, i.e.

$$\begin{aligned}\mathbf{T}\mathbf{G} &= \mathbf{G}\mathbf{T} = \mathbf{G}^{1/2}\mathbf{T}\mathbf{G}^{1/2}\\ \mathbf{T}_g\mathbf{G} &= \mathbf{G}\mathbf{T}_g = \mathbf{G}^{1/2}\mathbf{T}_g\mathbf{G}^{1/2},\end{aligned} \tag{52}$$

just as $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_g$ commute with $\mathbf{c}$. Also, with the introduction of the symmetric Piola–Kirchhoff stress tensor

$$\mathbf{S} = (\mathbf{F}^{-1})\boldsymbol{\tau}(\mathbf{F}^{-1})^T, \tag{53}$$

the dissipation inequality (9) takes the form

$$\begin{aligned}\dot{\gamma} &= \mathbf{S}\cdot\dot{\mathbf{E}} - \dot{\varphi} \geq 0\\ &= \left(\mathbf{S} - \frac{\partial\varphi}{\partial\mathbf{E}}\right)\cdot\dot{\mathbf{E}} - \frac{\partial\varphi}{\partial\mathbf{E}_p}\cdot\dot{\mathbf{E}}_p \geq 0,\end{aligned}$$

which in turn gives rise to the familiar results

$$\begin{aligned}\mathbf{S} &= \frac{\partial\varphi}{\partial\mathbf{E}}\\ \dot{\gamma} &= \mathbf{S}_p\cdot\dot{\mathbf{E}}_p \geq 0,\end{aligned} \tag{54}$$

in terms of the "thermodynamic tension"

$$\mathbf{S}_p \equiv -\frac{\partial\varphi}{\partial\mathbf{E}_p}. \tag{55}$$

Comparison of this dissipation rate expression with (31b) leads to the conclusion that

$$\begin{aligned}\mathbf{S}_p\cdot\dot{\mathbf{E}}_p &= \boldsymbol{\tau}\cdot\mathfrak{D}\\ &= \boldsymbol{\tau}\cdot[(\overline{\mathbf{F}}_p^{-1})^T\dot{\mathbf{E}}_p(\overline{\mathbf{F}}_p^{-1})]\\ &= [(\overline{\mathbf{F}}_p^{-1})\boldsymbol{\tau}(\overline{\mathbf{F}}_p^{-1})^T]\cdot\dot{\mathbf{E}}_p\\ \rightarrow \mathbf{S}_p &= (\overline{\mathbf{F}}_p^{-1})\boldsymbol{\tau}(\overline{\mathbf{F}}_p^{-1})^T.\end{aligned} \tag{56}$$

The stress expressions (53) and (56) together with (50c) then lead to the response equations

$$\begin{aligned}\mathbf{S} = \frac{\partial\varphi}{\partial\mathbf{E}} &= (\mathbf{F}^{-1})\boldsymbol{\tau}(\mathbf{F}^{-1})^T\\ &= (\mathbf{U}^{-1}\mathbf{R}^T)(\mathbf{R}\mathbf{T}\mathbf{R}^T)(\mathbf{R}\mathbf{U}^{-1})\\ &= \mathbf{U}^{-1}\mathbf{T}\mathbf{U}^{-1}\end{aligned} \tag{57a}$$

$$\begin{aligned}\mathbf{S}_p = -\frac{\partial\varphi}{\partial\mathbf{E}_p} &= (\overline{\mathbf{F}}_p^{-1})\boldsymbol{\tau}(\overline{\mathbf{F}}_p^{-1})^T\\ &= (\mathbf{U}^{-1}\mathbf{G}^{-1/2}\mathbf{R}^T)(\mathbf{R}\mathbf{T}\mathbf{R}^T)(\mathbf{R}\mathbf{G}^{-1/2}\mathbf{U}^{-1})\\ &= \mathbf{U}^{-1}\mathbf{T}\mathbf{G}^{-1}\mathbf{U}^{-1}.\end{aligned} \tag{57b}$$

Observe that these relationships make it possible to confirm the identity

$$\mathbf{S}\mathbf{C} = \mathbf{S}_p\mathbf{C}_p, \tag{58}$$

which was established in [3] as a necessary condition for the invariance of free energy under transformation of reference. By collecting these results and by making the appropriate replacements in the constitutive equations (33) and (34), the equivalent Lagrangian forms are realized, viz.

$$\varphi = \tilde{\varphi}(\mathbf{E}, \mathbf{E}_p) = \hat{\varphi}(\mathbf{G}), \qquad \mathbf{G} = \mathbf{U}^{-1}\mathbf{C}_p\mathbf{U}^{-1} \tag{59a}$$

$$\left.\begin{aligned}\mathbf{S} &= \frac{\partial\varphi}{\partial\mathbf{E}} = \mathbf{U}^{-1}\mathbf{T}\mathbf{U}^{-1}\\ \mathbf{S}_p &= -\frac{\partial\varphi}{\partial\mathbf{E}_p} = \mathbf{U}^{-1}\mathbf{T}\mathbf{G}^{-1}\mathbf{U}^{-1}\end{aligned}\right\}; \qquad \mathbf{T} = -2\left(\frac{\partial\varphi}{\partial\mathbf{G}}\,\mathbf{G}\right) \tag{59b}$$

$$\dot{\mathbf{E}}_p = \mathfrak{R}(\mathbf{E}, \mathbf{E}_p, \gamma, \dot{\mathbf{E}}) \equiv \mathbf{U}\mathbf{G}^{1/2}\hat{\mathfrak{D}}(\mathbf{G}, \gamma, \mathbf{U}^{-1}\dot{\mathbf{E}}\mathbf{U}^{-1})\mathbf{G}^{1/2}\mathbf{U} \tag{59c}$$

$$\dot{\gamma} = \mathbf{S}_p\cdot\dot{\mathbf{E}}_p \geq 0, \tag{59d}$$

where

$$\mathfrak{R} = 0, \quad \text{whenever} \begin{cases} g < 0 \text{ or} \\ g = 0 \text{ and } \mathbf{T}_g\cdot(\mathbf{U}^{-1}\dot{\mathbf{E}}\mathbf{U}^{-1}) \leq 0\end{cases} \tag{60}$$

and

$$\mathfrak{R} \neq 0 \text{ and } \dot{g} = 0, \quad \text{whenever } \begin{cases} g = 0 \text{ and} \\ \mathbf{T}_g \cdot (\mathbf{U}^{-1}\dot{\mathbf{E}}\mathbf{U}^{-1}) > 0, \end{cases}$$

in terms of

$$g = \hat{g}(\mathbf{E}, \mathbf{E}_p, \gamma) \equiv g(\mathbf{G}, \gamma)$$

and

$$\mathbf{T}_g \equiv -2\left(\frac{\partial g}{\partial \mathbf{G}}\mathbf{G}\right).$$

In view of the fact that the scalar invariants of the tensor

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{U}_p^{-1}\mathbf{C}\mathbf{U}_p^{-1} - \mathbf{I}) \tag{61}$$

are simply related to those of G, this material model clearly belongs to the special class considered by Naghdi and Trapp[6] (see eqns (37), (57)].

3.2. *The Lee connection*

The theory presented in [2]† is based on forms that are equivalent to

$$\begin{aligned} \varphi &= \bar{\varphi}(\mathbf{C}_e) \\ g &= f(\boldsymbol{\tau}, \gamma) = f(\acute{\boldsymbol{\tau}}, \gamma), \qquad \acute{\boldsymbol{\tau}} = \operatorname{dev}(\boldsymbol{\tau}) \\ \boldsymbol{\tau} &= 2\mathbf{F}_e \frac{\partial\bar{\varphi}}{\partial \mathbf{C}_e}\mathbf{F}_e^T \\ \mathfrak{D} &= \alpha\left(\frac{\partial f}{\partial \boldsymbol{\tau}}\right), \end{aligned} \tag{62}$$

for isotropic scalar energy and yield functions. Since

$$\varphi = \bar{\varphi}(\mathbf{C}_e) = \bar{\varphi}(\mathbf{R}_e^T\mathbf{b}\mathbf{R}_e) = \bar{\varphi}(\mathbf{b}) = \hat{\varphi}(\mathbf{c})$$

and

$$\begin{aligned} \boldsymbol{\tau} &= 2\mathbf{F}_e \frac{\partial\bar{\varphi}}{\partial \mathbf{C}_e}\mathbf{F}_e^T \\ &= 2(\mathbf{b}^{1/2}\mathbf{R}_e)\left(\mathbf{R}_e^T\frac{\partial\bar{\varphi}}{\partial \mathbf{b}}\mathbf{R}_e\right)(\mathbf{R}_e^T\mathbf{b}^{1/2}) \\ &= 2\mathbf{b}^{1/2}\left(\frac{\partial\bar{\varphi}}{\partial \mathbf{b}}\right)\mathbf{b}^{1/2} = 2\mathbf{b}\left(\frac{\partial\bar{\varphi}}{\partial \mathbf{b}}\right) \\ &= 2\mathbf{b}\left[-\mathbf{c}\left(\frac{\partial\hat{\varphi}}{\partial \mathbf{c}}\right)\mathbf{c}\right] \\ &= -2\left(\frac{\partial\varphi}{\partial \mathbf{c}}\mathbf{c}\right), \end{aligned}$$

† The fact that the theory proposed by Lee is a special case of the general theory of Green and Naghdi is established in [13].

it is evident that these response forms are equivalent to those of the present theory. The yield function and plastic deformation rate expression, however, are more specific than those appearing in (33) and (34). Lee's argument for these forms was based on certain physical considerations and the principle of maximum rate of plastic work. In the next section it is established that these forms are, in fact, necessary to guarantee material stability in the sense of Il'iushin[11].

4. IL'IUSHIN STABILITY

For a material to be Il'iushin stable, it is necessary and sufficient to guarantee that mechanical energy cannot be extracted from a material element during a closed deformation cycle. Put differently, Il'iushin stability requires that the energy dissipated must equal or exceed the internal energy released during any and all closed deformation cycles. In the context of the present development, this takes the form of the closed deformation path inequality

$$\Delta\gamma + \Delta\varphi = \int_{t_0}^{t_f} \boldsymbol{\tau}\cdot\mathbf{D}\, dt \geq 0. \tag{63}$$

The strictly dissipative nature of the plastic deformation mechanism is enforced by stipulating that the equality should hold only for purely elastic deformation cycles.

The restrictions imposed by this additional requirement can now be determined by making use of the general Il'iushin stability results as presented by Hill and Rice[12]. In the context of the present development, these restrictions (which are reviewed in Appendix A) take the form

$$[\delta_t^p \varphi]_{(\mathbf{E}_0, H_0, H_1)}^{(\mathbf{E}_1, H_0, \dot{H}_1)} = \frac{\partial}{\partial \mathbf{E}_p}[\varphi(\mathbf{E}_1, \mathbf{E}_p) - \varphi(\mathbf{E}_0, \mathbf{E}_p)]\cdot\dot{\mathbf{E}}_p \leq 0$$
$$\rightarrow [\mathbf{S}_p(\mathbf{E}_1, \mathbf{E}_p) - \mathbf{S}_p(\mathbf{E}_0, \mathbf{E}_p)]\cdot\dot{\mathbf{E}}_p \geq 0 \tag{64}$$

for all $\mathbf{E}_0 \in \{\mathbf{E}: g(\mathbf{E}, \mathbf{E}_p, \gamma) \leq 0\}$, and all plastic strain rates $\dot{\mathbf{E}}_p$ occurring at the transition state $\mathbf{E}_1 \in \{\mathbf{E}: g(\mathbf{E}, \mathbf{E}_p, \gamma) = 0\}$, and

$$\begin{aligned}(\delta_t^p \mathbf{S})\cdot\dot{\mathbf{E}} &= \dot{\mathbf{E}}\cdot[\partial\mathbf{S}/\partial\mathbf{E}_p]\cdot\dot{\mathbf{E}}_p \\ &= \dot{\mathbf{E}}\cdot\left[\frac{\partial^2\varphi}{\partial\mathbf{E}\otimes\partial\mathbf{E}_p}\right]\cdot\dot{\mathbf{E}}_p \\ &= -\dot{\mathbf{E}}_p\cdot[\partial\mathbf{S}_p/\partial\mathbf{E}]\cdot\dot{\mathbf{E}} \leq 0 \\ \rightarrow (\delta_t^e \mathbf{S}_p)\cdot\dot{\mathbf{E}}_p &= \left[\frac{d}{dt}\mathbf{S}_p\right]_{\mathbf{E}_p = \text{const.}}\cdot\dot{\mathbf{E}}_p \geq 0\end{aligned} \tag{65}$$

during plastic loading.

By making use of the plastic stress and deformation rate expressions (57b) and (48b), eqn (64) can be rewritten as

$$\begin{aligned}[\mathbf{S}_p(\mathbf{E}_1, \mathbf{E}_p) - \mathbf{S}_p(\mathbf{E}_0, \mathbf{E}_p)]\cdot\dot{\mathbf{E}}_p &\geq 0 \\ [(\overline{\mathbf{F}}_{p_1}^{-1})\boldsymbol{\tau}_1(\overline{\mathbf{F}}_{p_1}^{-1})^T - (\overline{\mathbf{F}}_{p_0}^{-1})\boldsymbol{\tau}_0(\overline{\mathbf{F}}_{p_0}^{-1})^T]\cdot[(\overline{\mathbf{F}}_{p_1})^T\mathcal{D}\overline{\mathbf{F}}_{p_1}] &\geq 0 \\ [\boldsymbol{\tau}_1 - (\overline{\mathbf{F}}_{p_1}\overline{\mathbf{F}}_{p_0}^{-1})\boldsymbol{\tau}_0(\overline{\mathbf{F}}_{p_1}\overline{\mathbf{F}}_{p_0}^{-1})^T]\cdot\mathcal{D} &\geq 0.\end{aligned} \tag{66}$$

In view of (42b), the plastic deformation gradients in this expression are given by

$$\overline{\mathbf{F}}_{p_0} = \mathbf{c}_0^{1/2}\mathbf{F}_0$$
$$\overline{\mathbf{F}}_{p_1} = \mathbf{c}_1^{1/2}\mathbf{F}_1,$$

where

$$\mathbf{F}_1 = \mathbf{F}\mathbf{F}_0$$

in terms of the deformation $\mathbf{F}$ accumulated during the elastic loading segment. As a consequence of (29), however, $\mathbf{F}$ has the representation

$$\mathbf{F} = \mathbf{b}_1^{1/2}\mathbf{Q}\mathbf{c}_0^{1/2}$$

in terms of some proper orthogonal $\mathbf{Q}$. It therefore follows that

$$\begin{aligned} \overline{\mathbf{F}}_{p1} &= \mathbf{c}_1^{1/2}(\mathbf{F}\mathbf{F}_0) \\ &= \mathbf{c}_1^{1/2}(\mathbf{b}_1^{1/2}\mathbf{Q}\mathbf{c}_0^{1/2})\mathbf{F}_0 \\ &= \mathbf{Q}\overline{\mathbf{F}}_{p0} \\ \rightarrow \overline{\mathbf{F}}_{p1}\overline{\mathbf{F}}_{p0}^{-1} &= \mathbf{Q} \end{aligned}$$

and consequently

$$(\boldsymbol{\tau}_1 - \mathbf{Q}\boldsymbol{\tau}_0\mathbf{Q}^T)\cdot\mathfrak{D} \geq 0. \tag{67}$$

The inequality (65) can be similarly reduced by making use of the fact that

$$\begin{aligned} &\left[\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{S}_p\right]_{\mathbf{E}_p=\text{const.}}\cdot\dot{\mathbf{E}}_p \geq 0 \\ &\left\{\frac{\mathrm{d}}{\mathrm{d}t}\,[(\overline{\mathbf{F}}_p^{-1})\boldsymbol{\tau}(\overline{\mathbf{F}}_p^{-1})^T]\right\}_{\mathbf{E}_p=\text{const.}}\cdot(\overline{\mathbf{F}}_p^T\mathfrak{D}\overline{\mathbf{F}}_p) \geq 0 \\ &\left[\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{U}_p^{-1}\overline{\mathbf{R}}_p^T\boldsymbol{\tau}\overline{\mathbf{R}}_p\mathbf{U}_p^{-1})\right]_{\mathbf{E}_p=\text{const.}}\cdot(\mathbf{U}_p\overline{\mathbf{R}}_p^T\mathfrak{D}\overline{\mathbf{R}}_p\mathbf{U}_p) \geq 0 \\ &\left\{\overline{\mathbf{R}}_p\left[\frac{\mathrm{d}}{\mathrm{d}t}\,(\overline{\mathbf{R}}_p^T\boldsymbol{\tau}\overline{\mathbf{R}}_p)\right]_{\mathfrak{D}=0}\overline{\mathbf{R}}_p^T\right\}\cdot\mathfrak{D} \geq 0 \\ &[\dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\overline{\mathbf{W}}_p - \overline{\mathbf{W}}_p\boldsymbol{\tau}]_{\mathfrak{D}=0}\cdot\mathfrak{D} \geq 0 \end{aligned} \tag{68}$$

in terms of the "unstretched" vorticity tensor

$$\overline{\mathbf{W}}_p = (\overline{\mathbf{L}}_p)_A = \dot{\overline{\mathbf{R}}}_p\overline{\mathbf{R}}_p^T$$

introduced in (46).

Expressed in terms of the stress and plastic deformation rate functions $\hat{\boldsymbol{\tau}}$ and $\hat{\mathfrak{D}}$ in (14), the inequality (67) is seen to require that

$$[\hat{\boldsymbol{\tau}}(\mathbf{c}_1) - \mathbf{Q}\hat{\boldsymbol{\tau}}(\mathbf{c}_0)\mathbf{Q}^T]\cdot\hat{\mathfrak{D}}(\mathbf{c}_1, \gamma, \mathbf{D}) \geq 0 \tag{69}$$

for any $\mathbf{c}_0 \in C(\gamma)$ and $\mathbf{c}_1 \in \partial C(\gamma)$, which are connected by the purely elastic deformation $\mathbf{F} = \mathbf{b}_1^{1/2}\mathbf{Q}\mathbf{c}_0^{1/2}$, and any deformation rate $\mathbf{D}$ at $t = t_1$. The strictly dissipative nature of plastic flow shall be enforced by stipulating that the equality should hold only for unloading or neutral loading, i.e. $\boldsymbol{\tau}_g\cdot\mathbf{D} \leq 0$.

Now, by again appealing to the discussion leading to (29) concerning the accessibility of elastic states in $C(\gamma)$, it is evident that there exist Il'iushin circuits that establish this same inequality for all $\mathbf{c}_1 \in \partial C(\gamma)$, all $\mathbf{c}_0 \in C(\gamma)$ and arbitrary proper orthogonal $\mathbf{Q}$—and in particular for $\mathbf{Q} = \mathbf{I}$. Thus, it is necessary to require that

$$[\hat{\boldsymbol{\tau}}(\mathbf{c}_1) - \hat{\boldsymbol{\tau}}(\mathbf{c}_0)]\cdot\hat{\mathfrak{D}}(\mathbf{c}_1, \gamma, \mathbf{D}) \geq 0 \tag{70}$$

for all $\mathbf{c}_1 \in \partial C(\gamma)$, all $\mathbf{c}_0 \in C(\gamma)$ and all deformation rates $\mathbf{D}$ at $t = t_1$. Observe that (70) guarantees (69) by virtue of (28) and the fact that $\hat{\boldsymbol{\tau}}$ is an isotropic function of $\mathbf{c}$. Note also that this inequality is identical in form to that obtained by Drucker[10].

To assess the implications of (70), the Kirchhoff stress function $\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}(\mathbf{c})$ is used to define the image of the elastic region in stress space, viz.

$$\mathbf{Y}(\gamma) \equiv \hat{\boldsymbol{\tau}}[C(\gamma)]. \tag{71}$$

The first implication of inequality (70) is that

$$\boldsymbol{\tau} \in \text{int } \mathbf{Y}(\gamma) \equiv \hat{\boldsymbol{\tau}}[\text{int } C(\gamma)]$$

if and only if

$$\boldsymbol{\tau} + p\mathbf{I} \in \text{int } \mathbf{Y}(\gamma)$$

for arbitrary real p. This follows since

$$[(\boldsymbol{\tau} + p\mathbf{I}) - \boldsymbol{\tau}]\cdot\mathfrak{D} = p \text{ tr}(\mathfrak{D}) = 0$$

for all possible plastic deformation rates due to the incompressibility constraint (18). It must be concluded, therefore, that it is impossible to reach an elastic-plastic transition state through purely hydrostatic loading. By continuity it must also follow that

$$\boldsymbol{\tau} \in \mathbf{Y}(\gamma)$$

if and only if

$$\boldsymbol{\tau} + p\mathbf{I} \in \mathbf{Y}(\gamma)$$

for all real p, and further that

$$\boldsymbol{\tau} \in \partial\mathbf{Y}(\gamma) \equiv \hat{\boldsymbol{\tau}}[\partial C(\gamma)]$$

if and only if

$$\boldsymbol{\tau} + p\mathbf{I} \in \partial\mathbf{Y}(\gamma)$$

for all real p. These conclusions may be conveniently rephrased in the form

$$\begin{aligned} \boldsymbol{\tau} \in \text{int } \mathbf{Y}(\gamma) \quad &\text{iff} \quad \acute{\boldsymbol{\tau}} \in \text{int } \mathbf{Y}(\gamma) \\ \boldsymbol{\tau} \in \partial\mathbf{Y}(\gamma) \quad &\text{iff} \quad \acute{\boldsymbol{\tau}} \in \partial\mathbf{Y}(\gamma), \end{aligned} \tag{72}$$

in terms of the Kirchhoff stress deviator

$$\acute{\boldsymbol{\tau}} = \text{dev}(\boldsymbol{\tau}) \equiv \boldsymbol{\tau} - \tfrac{1}{3}\text{ tr}(\boldsymbol{\tau})\mathbf{I}. \tag{73}$$

Thus, Il'iushin stability, through (70), guarantees a yield formulation in terms of the Kirchhoff stress deviator. That is, there must exist an isotropic scalar function h which implicitly defines the yield function (20) through the expressions

$$\begin{aligned} f(\boldsymbol{\tau}, \gamma) &= h(\acute{\boldsymbol{\tau}}, \gamma) \\ g(\mathbf{c}, \gamma) &= f[\hat{\boldsymbol{\tau}}(\mathbf{c}), \gamma]. \end{aligned} \tag{74}$$

With reference to (34), the yield criteria can now be recast in the alternative form

$$\mathfrak{D} = 0, \quad \text{whenever} \begin{cases} f(\tau, \gamma) < 0 \text{ or} \\ f(\tau, \gamma) = 0 \text{ and } \tau_g \cdot \mathbf{D} \leq 0, \end{cases}$$

and

$$\mathfrak{D} \neq 0 \text{ and } \dot{f} = \dot{g} = \tau_g \cdot (\mathbf{D} - \mathfrak{D}) + \frac{\partial f}{\partial \gamma} (\tau \cdot \mathfrak{D}) = 0,$$

$$\text{whenever} \begin{cases} f(\tau, \gamma) = 0 \text{ and} \\ \tau_g \cdot \mathbf{D} > 0 \end{cases} \tag{75}$$

in terms of

$$\tau_g = -2 \left(\frac{\partial g}{\partial \mathbf{c}} \mathbf{c} \right) = -2 \left[(\partial \hat{\tau} / \partial \mathbf{c})^T \cdot \frac{\partial f}{\partial \tau} \right] \mathbf{c}.$$

Moreover, after rewriting the Il'iushin and dissipation inequalities (70) and (31b) in their deviatoric form

$$[\acute{\tau}(t_1) - \acute{\tau}(t_0)] \cdot \mathfrak{D}(t_1) > 0$$

$$\acute{\tau}(t_1) \cdot \mathfrak{D}(t_1) > 0$$

whenever

$$\acute{\tau}(t_0) \in \Upsilon(\gamma)$$

$$\acute{\tau}(t_1) \in \partial \Upsilon(\gamma)$$

and

$$\mathfrak{D}(t_1) \neq 0,$$

it is evident that it is necessary and sufficient to require that the projection of $\Upsilon(\gamma)$ in deviatoric stress space,

$$\acute{\Upsilon}(\gamma) \equiv \{\tau: \tau \in \Upsilon(\gamma) \quad \text{and} \quad \mathrm{tr}(\tau) = 0\}, \tag{76}$$

form a convex neighborhood of the origin $\acute{\tau} = 0$, and that the plastic deformation rate at $\acute{\tau} \in \partial \acute{\Upsilon}(\gamma)$ be directed along the outward normal. After noting that

$$\begin{aligned} \mathrm{d}f = \frac{\partial f}{\partial \tau} \cdot \mathrm{d}\tau &= \mathrm{d}h \\ &= \frac{\partial h}{\partial \acute{\tau}} \cdot \mathrm{d}\acute{\tau} \\ &= \left(\frac{\partial h}{\partial \acute{\tau}} \right)' \cdot \mathrm{d}\acute{\tau} = \left(\frac{\partial h}{\partial \acute{\tau}} \right)' \cdot \mathrm{d}\tau, \end{aligned}$$

it follows that the normal to $\partial \acute{\Upsilon}(\gamma)$ in deviatoric, symmetric tensor space is given by

$$\left(\frac{\partial h}{\partial \acute{\tau}} \right)' = \frac{\partial f}{\partial \tau}.$$

Thus, the "normality" condition takes the familiar form

$$\mathfrak{D} = \alpha \left(\frac{\partial f}{\partial \boldsymbol{\tau}}\right), \tag{77}$$

in terms of a nonnegative scalar multiplier α. An immediate consequence of this is that $\mathfrak{D}$ shares principal directions, and thus commutes, with $\mathbf{c}$ so that

$$2\boldsymbol{\Gamma}_c = \mathbf{c}^{1/2} 2\mathfrak{D}\mathbf{c}^{1/2} = \mathbf{c}\mathfrak{D} + \mathfrak{D}\mathbf{c}. \tag{78}$$

This makes it possible to rewrite (33c) in the simpler form

$$\overset{\circ}{\mathbf{c}} = -(\mathbf{c}\mathbf{D}^* + \mathbf{D}^*\mathbf{c}), \qquad \mathbf{D}^* \equiv \mathbf{D} - \mathfrak{D} \tag{79}$$

in terms of an "effective" deformation rate $\mathbf{D}^*$. Moreover, since

$$\begin{aligned} \dot{f} = \dot{g} &= \boldsymbol{\tau}_g \cdot (\mathbf{D} - \mathfrak{D}) + \frac{\partial f}{\partial \gamma}(\boldsymbol{\tau} \cdot \mathfrak{D}) \\ &= \boldsymbol{\tau}_g \cdot \mathbf{D} - \left(\boldsymbol{\tau}_g - \frac{\partial f}{\partial \gamma}\boldsymbol{\tau}\right) \cdot \mathfrak{D} = 0 \end{aligned}$$

during plastic loading, it is clear that the scalar coefficient in (77) is given by

$$\alpha = \frac{\boldsymbol{\tau}_g \cdot \mathbf{D}}{\left(\boldsymbol{\tau}_g - \dfrac{\partial f}{\partial \gamma}\boldsymbol{\tau}\right) \cdot \dfrac{\partial f}{\partial \boldsymbol{\tau}}}. \tag{80}$$

This in turn gives rise to the additional requirement that

$$\left(\boldsymbol{\tau}_g - \frac{\partial f}{\partial \gamma}\boldsymbol{\tau}\right) \cdot \frac{\partial f}{\partial \boldsymbol{\tau}} > 0 \tag{81}$$

at all transition states by virtue of the fact that both α and $\boldsymbol{\tau}_g \cdot \mathbf{D}$ are positive during plastic loading.

Before collecting these results, observe that the second Il'iushin inequality, the plastic loading inequality (68), is now reduced to

$$\begin{aligned} [\dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\overline{\mathbf{W}}_p - \overline{\mathbf{W}}_p\boldsymbol{\tau}]_{\mathfrak{D}=0} \cdot \left(\alpha \frac{\partial f}{\partial \boldsymbol{\tau}}\right) &> 0 \\ [\dot{f}]_{\text{elastic}} + \left[\left(\boldsymbol{\tau}\frac{\partial f}{\partial \boldsymbol{\tau}} - \frac{\partial f}{\partial \boldsymbol{\tau}}\boldsymbol{\tau}\right) \cdot \overline{\mathbf{W}}_p\right]_{\mathfrak{D}=0} &> 0 \\ [\dot{f}]_{\text{elastic}} = [\dot{g}]_{\text{elastic}} &> 0 \\ \boldsymbol{\tau}_g \cdot \mathbf{D} &> 0 \end{aligned}$$

by virtue of the isotropic dependence of f on $\boldsymbol{\tau}$. Since this is precisely the plastic loading condition, no additional restrictions are obtained. Thus, the imposition of the Il'iushin inequalities (64) and (65) further restricts this theory to the form

$$\begin{aligned} \varphi &= \hat{\varphi}(\mathbf{c}), \qquad \rho = \rho_0[\det(\mathbf{c})]^{1/2} \\ \boldsymbol{\tau} &= \hat{\boldsymbol{\tau}}(\mathbf{c}) = -2\left(\frac{\partial \varphi}{\partial \mathbf{c}}\mathbf{c}\right) \\ \overset{\circ}{\mathbf{c}} &= \mathbf{c}(\mathfrak{D} - \mathbf{D}) + (\mathfrak{D} - \mathbf{D})\mathbf{c} \\ \dot{\gamma} &= \boldsymbol{\tau} \cdot \mathfrak{D} \\ \mathfrak{D} &= \alpha\left(\frac{\partial f}{\partial \boldsymbol{\tau}}\right), \end{aligned} \tag{82}$$

where

$$\alpha = 0, \text{ whenever } \begin{cases} f(\boldsymbol{\tau}, \gamma) < 0 \text{ or} \\ f(\boldsymbol{\tau}, \gamma) = 0 \text{ and } \boldsymbol{\tau}_g \cdot \mathbf{D} \leq 0, \end{cases}$$

and

$$\alpha = \frac{\boldsymbol{\tau}_g \cdot \mathbf{D}}{\left(\boldsymbol{\tau}_g - \frac{\partial f}{\partial \gamma} \boldsymbol{\tau}\right) \cdot \frac{\partial f}{\partial \boldsymbol{\tau}}}, \quad \text{whenever } \begin{cases} f(\boldsymbol{\tau}, \gamma) = 0 \text{ and} \\ \boldsymbol{\tau}_g \cdot \mathbf{D} > 0 \end{cases} \tag{83}$$

in terms of

$$\begin{aligned} f(\boldsymbol{\tau}, \gamma) &= h(\acute{\boldsymbol{\tau}}, \gamma) \\ g(\mathbf{c}, \gamma) &= f[\hat{\boldsymbol{\tau}}(\mathbf{c}), \gamma] \\ \boldsymbol{\tau}_g &= -2\left(\frac{\partial g}{\partial \mathbf{c}} \mathbf{c}\right). \end{aligned} \tag{84}$$

The specification of the scalar functions $\hat{\varphi}$ and h is subject to the requirement that $\mathbf{c} = \mathbf{I}$ represent a unique extremal point and global minimum for $\hat{\varphi}$, that $h(\acute{\boldsymbol{\tau}}, \gamma) < 0$ (for fixed γ) define a convex neighborhood of the origin $\acute{\boldsymbol{\tau}} = 0$ in stress deviator space and that inequality (81) be satisfied at transition states.

Recall that these forms, subject to all stated restrictions, guarantee that certain necessary conditions for Il'iushin stability are satisfied. Conceivably, additional restrictions attend more complicated deformation cycles. In this regard, it is also common to require that there exist infinitesimal stress cycles consisting of incremental plastic loading from a transition state followed by elastic stress recovery. It is clear that the Il'iushin inequality is automatically satisfied for such a path since $\Delta\varphi = 0$ and $\Delta\gamma > 0$. However, for such paths to exist in the first place, it is necessary to require that the material "stress harden" during plastic flow in the sense that

$$\frac{\partial f}{\partial \gamma} < 0, \quad \text{whenever} \quad f(\boldsymbol{\tau}, \gamma) = 0. \tag{85}$$

In the final section, it is established that an elliptical energy function renders this restriction more severe than (81), and thus the possibility of a strain-softening material is not precluded by Il'iushin stability. Moreover, since

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\tau}} \cdot \mathring{\boldsymbol{\tau}} + \frac{\partial f}{\partial \gamma} \dot{\gamma},$$

it follows that

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\tau}} \cdot \mathring{\boldsymbol{\tau}} \leq 0$$

for neutral loading or unloading ($\dot{\gamma} = 0$) from a yield point ($f = 0$), and that

$$\begin{aligned} \dot{f} = \frac{\partial f}{\partial \boldsymbol{\tau}} \cdot \mathring{\boldsymbol{\tau}} + \frac{\partial f}{\partial \gamma} \dot{\gamma} &= 0 \\ \rightarrow \frac{\partial f}{\partial \boldsymbol{\tau}} \cdot \mathring{\boldsymbol{\tau}} &= -\frac{\partial f}{\partial \gamma} \dot{\gamma} \end{aligned}$$

during plastic loading. In view of the fact that $\dot{\gamma} > 0$ during plastic loading, it immediately follows that the additional "hardening" restriction $\partial f/\partial\gamma < 0$ is necessary and sufficient to establish the correspondence

$$\operatorname{sgn}(\boldsymbol{\tau}_g \cdot \mathbf{D}) = \operatorname{sgn}\left(\frac{\partial f}{\partial \boldsymbol{\tau}} \cdot \mathring{\boldsymbol{\tau}}\right) \tag{86}$$

whenever $f = 0$. This result could then be used to restate the yield criteria (83) in a more familiar "stress-based" form that would prove useful for stress-controlled (as opposed to deformation-controlled) processes.

Before proceeding, it is also worthwhile to confirm that the identities

$$\begin{aligned} f(\mathbf{T}, \gamma) &= f(\boldsymbol{\tau}, \gamma) \\ \frac{\partial f}{\partial \mathbf{T}} &= \mathbf{R}^T \frac{\partial f}{\partial \boldsymbol{\tau}} \mathbf{R} \\ \mathbf{T}_g \cdot \frac{\partial f}{\partial \mathbf{T}} &= \boldsymbol{\tau}_g \cdot \frac{\partial f}{\partial \boldsymbol{\tau}} \\ \mathbf{T} \cdot \frac{\partial f}{\partial \mathbf{T}} &= \boldsymbol{\tau} \cdot \frac{\partial f}{\partial \boldsymbol{\tau}}, \end{aligned} \tag{87}$$

and

$$\begin{aligned} \hat{\mathfrak{D}}(\mathbf{G}, \gamma, \mathbf{U}^{-1}\dot{\mathbf{E}}\mathbf{U}^{-1}) &= \hat{\mathfrak{D}}(\mathbf{R}^T\mathbf{c}\mathbf{R}, \gamma, \mathbf{R}^T\mathbf{D}\mathbf{R}) \\ &= \mathbf{R}^T\hat{\mathfrak{D}}(\mathbf{c}, \gamma, \mathbf{D})\mathbf{R} \\ &= \alpha\mathbf{R}^T\left(\frac{\partial f}{\partial \boldsymbol{\tau}}\right)\mathbf{R} \\ &= \alpha\left(\frac{\partial f}{\partial \mathbf{T}}\right), \end{aligned} \tag{88}$$

follow as a consequence of (40, 48c, 50). Thus, the plastic rate form (59c) can now be rewritten as

$$\dot{\mathbf{E}}_p = \alpha\mathbf{U}\mathbf{G}^{1/2}\frac{\partial f}{\partial \mathbf{T}}\mathbf{G}^{1/2}\mathbf{U},$$

where

$$\alpha = \frac{\mathbf{U}^{-1}\mathbf{T}_g\mathbf{U}^{-1}}{\left(\mathbf{T}_g - \frac{\partial f}{\partial \mathbf{T}}\mathbf{T}\right)\cdot\frac{\partial f}{\partial \mathbf{T}}}\cdot\dot{\mathbf{E}}, \quad \text{whenever} \begin{cases} f(\mathbf{T}, \gamma) = 0 \text{ and} \\ \mathbf{T}_g \cdot \mathbf{U}^{-1}\dot{\mathbf{E}}\mathbf{U}^{-1} > 0, \end{cases} \tag{89}$$

and

$$\alpha = 0, \qquad \text{otherwise.}$$

It seems that it should also be possible to derive these or equivalent Lagrangian rate forms by applying the general "work postulate" inequalities of Naghdi and Trapp[7, 14] directly to the special forms given in (59) and (60). However, this process would be impeded by the (algebraically) complicated nature of the yield function transformation

$$\hat{f}(\mathbf{S}; \mathbf{E}_p, \gamma) = f[\hat{\mathbf{T}}(\mathbf{S}, \mathbf{E}_p), \gamma], \tag{90}$$

where

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{S}, \mathbf{E}_p) = \hat{\mathbf{U}}(\mathbf{S}, \mathbf{E}_p)\mathbf{S}\hat{\mathbf{U}}(\mathbf{S}, \mathbf{E}_p)$$

in terms of the "total stretch" function

$$\mathbf{U} = \hat{\mathbf{U}}(\mathbf{S}, \mathbf{E}_p).$$

In any event, comparison of the Lagrangian forms (59) and (60) and (89) with the equivalent Eulerian forms (82) and (83) lends credence to the assertion that this particular theory is better suited to an Eulerian formulation. The relevance of this observation derives from the relative importance of this particular (highly specialized) material model—based as it is on widely understood and historically important material hypotheses.

A final alternative form for (59c) can be written in terms of the nonsymmetric stress measure

$$\begin{aligned} \boldsymbol{\Sigma} &\equiv \mathbf{SC} \\ &= (\mathbf{U}^{-1}\mathbf{T}\mathbf{U}^{-1})\mathbf{U}^2 \\ &= \mathbf{U}^{-1}\mathbf{T}\mathbf{U}. \end{aligned} \tag{91}$$

Since the invariants of $\boldsymbol{\Sigma}$ are identical to the invariants of $\mathbf{T}$, it is clear that

$$f(\boldsymbol{\Sigma}, \gamma) = f(\mathbf{T}, \gamma) = f(\boldsymbol{\tau}, \gamma), \tag{92}$$

and

$$\frac{\partial f}{\partial \boldsymbol{\Sigma}} = \mathbf{U}^{-1}\frac{\partial f}{\partial \mathbf{T}}\mathbf{U}.$$

Owing to the commutative property (52), it then follows that

$$\begin{aligned} \dot{\mathbf{E}}_p &= \alpha\mathbf{UG}\frac{\partial f}{\partial \mathbf{T}}\mathbf{U} \\ &= \alpha(\mathbf{UGU})\left(\mathbf{U}^{-1}\frac{\partial f}{\partial \mathbf{T}}\mathbf{U}\right) \\ &= \alpha\mathbf{C}_p\frac{\partial f}{\partial \boldsymbol{\Sigma}} \\ \rightarrow \mathbf{B}_p\dot{\mathbf{E}}_p &= \alpha\left(\frac{\partial f}{\partial \boldsymbol{\Sigma}}\right). \end{aligned} \tag{93}$$

This confirms the plastic rate form suggested in [3].

An illustration of the usefulness of this final form involves the J_2-type yield function

$$f(\boldsymbol{\tau}, \gamma) = \tfrac{1}{2}\boldsymbol{\tau}'\cdot\boldsymbol{\tau}' - \kappa(\gamma),$$

for which

$$\begin{aligned} \frac{\partial f}{\partial \boldsymbol{\tau}} &= \boldsymbol{\tau}' = \boldsymbol{\tau} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\tau})\mathbf{I} \\ \rightarrow \frac{\partial f}{\partial \mathbf{T}} &= \mathbf{R}^T\frac{\partial f}{\partial \mathbf{T}}\mathbf{R} = \mathbf{T} - \frac{1}{3}\operatorname{tr}(\mathbf{T})\mathbf{I} \\ \rightarrow \frac{\partial f}{\partial \boldsymbol{\Sigma}} &= \mathbf{U}^{-1}\frac{\partial f}{\partial \mathbf{T}}\mathbf{U} = \boldsymbol{\Sigma} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\Sigma})\mathbf{I}. \end{aligned}$$

In view of the identity (58), it immediately follows from (93) that

$$\dot{\mathbf{E}}_p = \alpha \mathbf{C}_p[\mathbf{S}_p - \tfrac{1}{3}\mathbf{B}_p(\mathbf{C}_p \cdot \mathbf{S}_p)]\mathbf{C}_p,$$

in terms of the thermodynamic tension $\mathbf{S}_p$.

5. LOG STRAIN AND SMALL STRAIN FORMULATION

In this section, the constitutive forms (82) and (83) are recast through the replacement of the elastic deformation tensor $\mathbf{c}$ with the elastic log-strain tensor $\mathbf{e}$ defined through the expression

$$\begin{aligned} 2\mathbf{e} &\equiv -\ln(\mathbf{c}) \\ &= \sum_{n=1}^{\infty} \frac{1}{n}(\mathbf{I} - \mathbf{c})^n, \qquad \|\mathbf{I} - \mathbf{c}\| < 1 \\ \rightarrow \mathbf{c} &= \exp(-2\mathbf{e}) = \sum_{n=0}^{\infty} \frac{1}{n!}(-2\mathbf{e})^n. \end{aligned} \tag{94}$$

The specific properties of this tensor transformation are developed in Appendix B. For present purposes, it is important to recall that

$$\mathbf{e} = 0 \quad \text{iff} \quad \mathbf{c} = \mathbf{I}, \tag{95}$$

that

$$\mathrm{tr}(\mathbf{e}) = \ln([\det(\mathbf{c})]^{-1/2}) \tag{96}$$

and that chain rule expansion is accomplished through the expressions

$$\begin{aligned} \frac{d\mathbf{c}}{d\eta} &= \left(\frac{\partial \mathbf{c}}{\partial \mathbf{e}}\right) \cdot \frac{d\mathbf{e}}{d\eta}; \qquad \frac{\partial \varphi}{\partial \mathbf{e}} = \left(\frac{\partial \mathbf{c}}{\partial \mathbf{e}}\right) \cdot \frac{\partial \varphi}{\partial \mathbf{c}} \\ \frac{d\mathbf{e}}{d\eta} &= \left(\frac{\partial \mathbf{e}}{\partial \mathbf{c}}\right) \cdot \frac{d\mathbf{c}}{d\eta}, \qquad \frac{\partial \varphi}{\partial \mathbf{c}} = \left(\frac{\partial \mathbf{e}}{\partial \mathbf{c}}\right) \cdot \frac{\partial \varphi}{\partial \mathbf{e}}, \end{aligned} \tag{97}$$

which are written in terms of the symmetric "double tensors"

$$\begin{aligned} \left(\frac{\partial \mathbf{c}}{\partial \mathbf{e}}\right) \cdot \mathbf{A} &= \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} \left\{ \sum_{\kappa=1}^{n} \mathbf{e}^{n-\kappa}\mathbf{A}\mathbf{e}^{\kappa-1} \right\} \\ &= -2\mathbf{c}\mathbf{A}, \text{ whenever } \mathbf{A}\mathbf{c} = \mathbf{c}\mathbf{A} \\ \left(\frac{\partial \mathbf{e}}{\partial \mathbf{c}}\right) \cdot \mathbf{A} &= -\frac{1}{2}\left\{ \sum_{n=1}^{\infty} \frac{1}{n} \left[\sum_{\kappa=1}^{n} (\mathbf{I} - \mathbf{c})^{n-\kappa}\mathbf{A}(\mathbf{I} - \mathbf{c})^{\kappa-1} \right] \right\} \\ &= -\tfrac{1}{2}\mathbf{b}\mathbf{A}, \text{ whenever } \mathbf{A}\mathbf{c} = \mathbf{c}\mathbf{A}. \end{aligned} \tag{98}$$

Use shall also be made of the "small strain" expansion

$$\begin{aligned} \left(\frac{\partial \mathbf{e}}{\partial \mathbf{c}}\right) \cdot (\mathbf{c}\mathbf{B} + \mathbf{B}\mathbf{c}) &= -\{\mathbf{B} + \tfrac{1}{3}[\mathbf{e}^2\mathbf{B} - 2\mathbf{e}\mathbf{B}\mathbf{e} + \mathbf{B}\mathbf{e}^2] + \ldots\} \\ &= -\mathbf{B}, \text{ whenever } \mathbf{B}\mathbf{c} = \mathbf{c}\mathbf{B}. \end{aligned} \tag{99}$$

Now, since the energy and yield functions are isotropic functions of their tensor argument $\mathbf{c}$, it follows from (97) and (98) that

$$\begin{aligned}\boldsymbol{\tau} &= -2\left(\frac{\partial \varphi}{\partial \mathbf{c}}\mathbf{c}\right) = -2\left[\left(\frac{\partial \mathbf{e}}{\partial \mathbf{c}}\right)\cdot\frac{\partial \varphi}{\partial \mathbf{e}}\right]\mathbf{c} \\ &= -2\left(-\frac{1}{2}\mathbf{b}\frac{\partial \varphi}{\partial \mathbf{e}}\right)\mathbf{c} \\ &= \frac{\partial \varphi}{\partial \mathbf{e}},\end{aligned} \tag{100}$$

and

$$\boldsymbol{\tau}_g = -2\left(\frac{\partial g}{\partial \mathbf{c}}\mathbf{c}\right) = \frac{\partial g}{\partial \mathbf{e}}.$$

In view of (74), it then follows that

$$\begin{aligned}\boldsymbol{\tau}_g &= \frac{\partial g}{\partial \mathbf{e}} \\ &= \frac{\partial f}{\partial \boldsymbol{\tau}}\cdot\left(\frac{\partial \boldsymbol{\tau}}{\partial \mathbf{e}}\right) \\ &= \underset{\sim}{\mathbf{K}}\cdot\frac{\partial f}{\partial \boldsymbol{\tau}}\end{aligned} \tag{101}$$

in terms of the symmetric, fourth-order ''ellipticity'' tensor

$$\underset{\sim}{\mathbf{K}} \equiv \left(\frac{\partial \boldsymbol{\tau}}{\partial \mathbf{e}}\right) = \frac{\partial^2 \varphi}{\partial \mathbf{e}\otimes\partial \mathbf{e}}. \tag{102}$$

Finally, the evolution equation for $\mathbf{e}$, viz.

$$\begin{aligned}\mathring{\mathbf{e}} &= \left(\frac{\partial \mathbf{e}}{\partial \mathbf{c}}\right)\cdot\mathring{\mathbf{c}} \\ &= \left(\frac{\partial \mathbf{e}}{\partial \mathbf{c}}\right)\cdot[\mathbf{c}(\mathfrak{D} - \mathbf{D}) + (\mathfrak{D} - \mathbf{D})\mathbf{c}] \\ &= -\left(\frac{\partial \mathbf{e}}{\partial \mathbf{c}}\right)\cdot(\mathbf{cD} + \mathbf{Dc}) - \mathfrak{D} \\ &= \mathbf{D} - \mathfrak{D}, \text{ whenever } \mathbf{cD} = \mathbf{Dc},\end{aligned} \tag{103}$$

follows as a consequence of (97) and (98) and the commutative property (78). For small elastic strain, expansion (99) may be used to obtain

$$\mathring{\mathbf{e}} = \mathbf{D} - \mathfrak{D} + \tfrac{1}{3}(\mathbf{e}^2\mathbf{D} - 2\mathbf{eDe} + \mathbf{De}^2) + \ldots. \tag{104}$$

After collecting results and making use of the appropriate substitutions in (82) and (83), the equivalent log-strain forms are obtained, viz.

$$
\begin{aligned}
\varphi &= \hat{\varphi}(\mathbf{e}), \qquad \ln\left(\frac{\rho_0}{\rho}\right) = \mathrm{tr}(\mathbf{e}) \\
f(\boldsymbol{\tau}, \gamma) &= h(\acute{\boldsymbol{\tau}}, \gamma), \qquad \acute{\boldsymbol{\tau}} = \boldsymbol{\tau} - \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\tau})\mathbf{I} \\
\boldsymbol{\tau} &= \frac{\partial \varphi}{\partial \mathbf{e}} \\
\mathring{\mathbf{e}} &= \begin{cases} -\left(\dfrac{\partial \mathbf{e}}{\partial \mathbf{c}}\right) \cdot (\mathbf{cD} + \mathbf{Dc}) - \mathfrak{D} \\ \mathbf{D} - \mathfrak{D} + \tfrac{1}{3}[\mathbf{e}^2\mathbf{D} - 2\mathbf{eDe} + \mathbf{De}^2] + \dots \\ \mathbf{D} - \mathfrak{D}, \text{ whenever } \mathbf{cD} = \mathbf{Dc} \end{cases} \\
\dot{\gamma} &= \boldsymbol{\tau} \cdot \mathfrak{D} \\
\mathfrak{D} &= \alpha\left(\frac{\partial f}{\partial \boldsymbol{\tau}}\right),
\end{aligned}
\tag{105}
$$

where

$$
\alpha = \frac{\underset{\sim}{\mathbf{K}} \cdot \partial f/\partial \boldsymbol{\tau}}{\partial f/\partial \boldsymbol{\tau} \cdot \underset{\sim}{\mathbf{K}} \cdot \partial f/\partial \boldsymbol{\tau} - \partial f/\partial \gamma(\boldsymbol{\tau} \cdot \partial f/\partial \boldsymbol{\tau})} \cdot \mathbf{D}, \text{ whenever } \begin{cases} f(\boldsymbol{\tau}, \gamma) = 0 \text{ and} \\ \mathbf{D} \cdot \underset{\sim}{\mathbf{K}} \cdot \partial f/\partial \boldsymbol{\tau} > 0, \end{cases} \tag{106}
$$

and

$$
\alpha = 0, \text{ otherwise.}
$$

As before, $\mathbf{e} = 0$ must represent a unique extremal point and a global minimum for $\hat{\varphi}$, $h(\acute{\boldsymbol{\tau}}, \gamma) < 0$ (for fixed γ) must define a convex neighborhood of the origin in stress deviator space and

$$
\frac{\partial f}{\partial \gamma} < \frac{\partial f/\partial \boldsymbol{\tau} \cdot \underset{\sim}{\mathbf{K}} \cdot \partial f/\partial \boldsymbol{\tau}}{(\boldsymbol{\tau} \cdot \partial f/\partial \boldsymbol{\tau})}, \text{ whenever } f(\boldsymbol{\tau}, \gamma) = 0 \tag{107}
$$

to ensure that $\alpha > 0$ during plastic loading. As claimed in the previous section, it is now clear that an elliptic energy function, characterized by positive definite $\underset{\sim}{\mathbf{K}}$, guarantees that this condition is satisfied whenever

$$
\frac{\partial f}{\partial \gamma} < 0
$$

at transition states.

These forms are particularly useful in the event that the elastic strain is "small". In this case, the elastic strain evolution equation

$$
\mathring{\mathbf{e}} = \mathbf{D} - \mathfrak{D}, \tag{108}
$$

which is exact whenever $\mathbf{D}$ and $\mathbf{e}$ share principal directions, is accurate to terms of order $\|\mathbf{D}\|\,\|\mathbf{e}\|^2$. It is important to note that the assumption of small elastic strain in no way implies "small deformation", since no restriction is imposed on the amount of plastic deformation that may accumulate. Thus, the log-strain equations (105), with the linearized rate form (108), may well provide the basis for the modeling of many large deformation-forming processes.

Of further interest is the fact that the most general polynomial expansion for the isotropic energy function, satisfying all stated restrictions, has the form

$$\hat{\varphi}(e) = \varphi_0 + \tfrac{1}{2}e\cdot \underset{\sim}{K}_0\cdot e + \ldots$$
$$= \varphi_0 + \tfrac{1}{2}[\lambda\ \mathrm{tr}^2(e) + 2\mu e\cdot e] + \ldots . \tag{109}$$

This provides rational justification for the fully linearized forms

$$\begin{aligned} \tau &= \lambda\ \mathrm{tr}(e)\mathbf{I} + 2\mu e \\ \mathring{e} &= \mathbf{D} - \mathfrak{D} \\ \dot{\gamma} &= \tau\cdot\mathfrak{D} \\ \mathfrak{D} &= \alpha\left(\frac{\partial f}{\partial \tau}\right), \end{aligned} \tag{110}$$

with

$$\alpha = \frac{\partial f/\partial\tau\cdot\mathbf{D}}{\dfrac{\partial f}{\partial \tau}\cdot\dfrac{\partial f}{\partial \tau} - \dfrac{1}{2\mu}\dfrac{\partial f}{\partial \gamma}\left(\tau\cdot\dfrac{\partial f}{\partial \tau}\right)}, \text{ whenever } \begin{cases} f(\tau, \gamma) = 0 \text{ and} \\ \partial f/\partial\tau\cdot\mathbf{D} > 0 \end{cases} \tag{111}$$

and

$$\alpha = 0, \text{ otherwise.}$$

These forms, which accurately approximate the exact theory in the small elastic strain regime, are easily recognized as a corotational form of the classical Prandtl–Reuss equations. For completeness note, once again, that the yield function f must define a convex neighborhood of the origin in $\mathring{\tau}$-space and that the inequality (81) is guaranteed by imposing the "hardening" requirement $\partial f/\partial\gamma < 0$. With reference to (85) and (86), it is also clear that this latter "overstrict" condition would make it possible to restate the plastic loading requirements in the equivalent form

$$f(\tau, \gamma) = 0, \qquad \frac{\partial f}{\partial \tau}\cdot\mathring{\tau} > 0. \tag{112}$$

Elimination of e from the small strain forms (110) is easily achieved by differentiating the stress response equation and making the appropriate strain rate substitutions. This leads to the rate-independent, symmetric, hypoelastic form

$$\mathring{\tau} = \begin{cases} 2\left[\underset{\sim}{K}_0 - \mu\beta\left(\dfrac{\partial f}{\partial \tau}\otimes\dfrac{\partial f}{\partial \tau}\right)\right]\cdot\mathbf{D} \\ \dfrac{\partial}{\partial \mathbf{D}}\left\{\mathbf{D}\cdot\left[\underset{\sim}{K}_0 - \mu\beta\left(\dfrac{\partial f}{\partial \tau}\otimes\dfrac{\partial f}{\partial \tau}\right)\right]\cdot\mathbf{D}\right\}, \end{cases} \tag{113}$$

where

$$\beta = \left[\frac{\partial f}{\partial \tau}\cdot\frac{\partial f}{\partial \tau} - \frac{1}{2\mu}\frac{\partial f}{\partial \gamma}\left(\tau\cdot\frac{\partial f}{\partial \tau}\right)\right]^{-1}, \text{ whenever } \begin{cases} f = 0 \text{ and} \\ \partial f/\partial\tau\cdot\mathbf{D} > 0, \end{cases} \tag{114}$$

and

$$\beta = 0, \text{ otherwise.}$$

This particular form, employing Cauchy rather than Kirchhoff stress in the yield formulation, has already been used by McMeeking and Rice[15] and others to model large deformation elastoplastic phenomena. It is interesting to note that the popularity of this model is evidently based on the fact that it leads to a symmetric stiffness matrix in a finite element formulation—in spite of misgivings regarding its suitability as a material model. Through this final exercise, it is now apparent that in comparison with other Prandtl–Reuss generalizations, this form (in addition to being the most convenient) most accurately approximates the exact material model.

These final considerations offer an alternative to those who hold that a small deformation incremental theory can be made "valid" for large deformation merely by rephrasing it in terms of "properly invariant" time derivatives. Such an approach affords no understanding of the extended circumstances under which the generalized mathematical model continues to accurately represent the original material hypotheses. The present development suggests that the formulation of large deformation theory should be based on strict mathematical interpretation of the material hypotheses and a precise accounting for geometry change from the outset. Such a theory, if not immediately "usable" in its own right, will at least provide a rational context for the selection of an approximate theory. The large strain state-variable format detailed in [9] establishes a theoretical foundation that is particularly well suited to this sort of approach.

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APPENDIX A

In terms of the symmetric Piola–Kirchhoff stress $\mathbf{S}$ and its conjugate the total Green strain $\mathbf{E}$, the dissipation (per unit reference volume) inequality is given by

$$\dot{\gamma} = \mathbf{S}\cdot\dot{\mathbf{E}} - \dot{\varphi} \geq 0, \tag{A1}$$

while the Il'iushin inequality is expressed as

$$\int_{t0}^{tf} \mathbf{S}\cdot\dot{\mathbf{E}}\,\mathrm{d}t = \Delta\gamma + \Delta\varphi \geq 0 \tag{A2}$$

for any and all closed deformation paths

$$\mathbf{E} = \mathbf{E}(t), \qquad t_0 \leq t \leq t_f, \text{ with } \mathbf{E}(t_0) = \mathbf{E}(t_f). \tag{A3}$$

Grouping all inelastic Lagrangian variables under the collective symbol H, the established constitutive forms

$$\begin{aligned} \varphi &= \hat{\varphi}(\mathbf{E}, H) \\ \mathbf{S} &= \hat{\mathbf{S}}(\mathbf{E}, H), \end{aligned} \tag{A4}$$

and the notational shorthand

$$Z = Z(\mathbf{E}, H) \begin{cases} Z_{H0}(\mathbf{E}) \equiv Z(\mathbf{E}, H_0) \\ \delta_t^e Z \equiv \left(\dfrac{\partial Z}{\partial \mathbf{E}}\right) \cdot \dot{\mathbf{E}} \\ \delta_t^p Z \equiv \left(\dfrac{\partial Z}{\partial H}\right) \cdot \dot{H} \end{cases} \tag{A5}$$

shall now be used to expand the Il'iushin inequality (A2). With reference to Fig. 1, this inequality shall be enforced along a closed deformation path consisting of:

(i) an initial elastic segment leading from $(\mathbf{E}_0, H_0) \to (\mathbf{E}_1, H_0)$, with

$$\mathbf{E}_0 \in \mathscr{E}(H_0) \equiv \{\mathbf{E}: f(\mathbf{E}, H_0) \le 0\}$$

and (A6)

$$\mathbf{E}_1 \in \partial\mathscr{E}(H_0) \equiv \{\mathbf{E}: f(\mathbf{E}, H_0) = 0\},$$

in terms of a yield function f; followed by

(ii) a plastic loading segment from $(\mathbf{E}_1, H_0) \to (\mathbf{E}_1 + \Delta\mathbf{E}, H_0 + \Delta H)$; and concluding with

(iii) a terminal segment of elastic unloading leading from $(\mathbf{E} + \Delta E, H_0 + \Delta H) \to (\mathbf{E}_0, H_0 + \Delta H)$ with $\mathbf{E}_0 \in \mathscr{E}(H_0 + \Delta H)$.

Since $\dot{\gamma} = 0$ and $H =$ const. on each of the elastic segments, (A2) takes the form:

$$\begin{aligned} \int_{t0}^{tf} \mathbf{S}\cdot\dot{\mathbf{E}}\,\mathrm{d}t &= \int_{t0}^{tf} (\dot{\varphi} + \dot{\gamma})\,\mathrm{d}t \ge 0 \\ \int_{t0}^{t1} \dot{\varphi}_{H0}\,\mathrm{d}t + \int_{t1}^{t1+\Delta t} \mathbf{S}\cdot\dot{\mathbf{E}}\,\mathrm{d}t &+ \int_{t1+\Delta t}^{tf} \dot{\varphi}_{H0+\Delta H}\,\mathrm{d}t \ge 0 \\ [\varphi_{H0}]_{\mathbf{E}0}^{\mathbf{E}1} + \int_{t1}^{t1+\Delta t} (\mathbf{S} - \mathbf{S}_{H0+\Delta H})\cdot\dot{\mathbf{E}}\,\mathrm{d}t &+ \int_{t1}^{t1+\Delta t} \mathbf{S}_{H0+\Delta H}\cdot\dot{\mathbf{E}}\,\mathrm{d}t + [\varphi_{H0+\Delta H}]_{\mathbf{E}1+\Delta\mathbf{E}}^{\mathbf{E}0} \ge 0. \end{aligned} \tag{A7}$$

But

$$\mathbf{S}_{H0+\Delta H}\cdot\dot{\mathbf{E}} = \dot{\varphi}_{H0+\Delta H} \to \int_{t1}^{t1+\Delta t} \mathbf{S}_{H0+\Delta H}\cdot\dot{\mathbf{E}}\,\mathrm{d}t = [\varphi_{H0+\Delta H}]_{\mathbf{E}1}^{\mathbf{E}1+\Delta\mathbf{E}}, \tag{A8}$$

so that

$$\begin{aligned} &\int_{t0}^{tf} \mathbf{S}\cdot\dot{\mathbf{E}}\,\mathrm{d}t = -\left\{[\varphi_{H0+\Delta H} - \varphi_{H0}]_{\mathbf{E}0}^{\mathbf{E}1} + \int_{t1}^{t1+\Delta t} (\mathbf{S}_{H0+\Delta H} - \mathbf{S})\cdot\dot{\mathbf{E}}\,\mathrm{d}t\right\} \ge 0 \\ &\quad\to [\varphi_{H0+\Delta H} - \varphi_{H0}]_{\mathbf{E}0}^{\mathbf{E}1} + \{[\hat{\mathbf{S}}(\mathbf{E}, H_0 + \Delta H) - \hat{\mathbf{S}}(\mathbf{E}, H)]\cdot\dot{\mathbf{E}}\}_{t=t^*}\Delta t \le 0, \qquad t_1 \le t^* \le (t_1 + \Delta t), \end{aligned} \tag{A9}$$

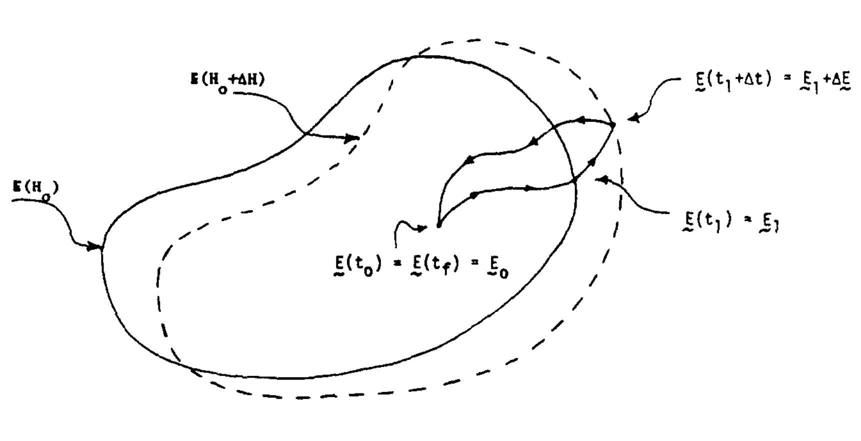

Fig. 1. Closed isothermal strain cycle.

where the mean value theorem has been used to evaluate the integral term. Assuming smooth dependence of $\hat{\varphi}$ and $\hat{\mathbf{S}}$ on H and smooth variation of H during the inelastic segment, it follows that

$$\varphi_{H_0+\Delta H} - \varphi_{H_0} = [(\delta_t^p\varphi) + O(\Delta t)]\Delta t$$

and (A10)

$$\hat{\mathbf{S}}(\mathbf{E}^*, H_0 + \Delta H) - \hat{\mathbf{S}}(\mathbf{E}^*, H^*) = [(\delta_t^p\mathbf{S}) + O(\Delta t^*)]_{t=t^*}\Delta t^*,$$

with

$$\Delta t^* \equiv (t_1 + \Delta t) - t^* \to 0 \leq \Delta t^* \leq \Delta t. \tag{A11}$$

Thus, to lowest-order terms, the inequality takes the final form

$$[(\delta_t^p\varphi) + O(\Delta t)]_{\mathbf{E}_0}^{\mathbf{E}_1} + \{[(\delta_t^p\mathbf{S}) + O(\Delta t)]\cdot\dot{\mathbf{E}}\}_{t=t^*}\Delta t^* \leq 0,$$

where $t_1 \leq t^* \leq (t_1 + \Delta t)$ and $\Delta t^* = (t_1 + \Delta t) - t^* \leq \Delta t$. (A12)

Since it is possible to choose an Il'iushin circuit for which $\mathbf{E}_0 \neq \mathbf{E}_1$ and $\Delta t \ll 1$, it is clearly necessary to require that

$$[\delta_t^p\varphi]_{\mathbf{E}_0}^{\mathbf{E}_1} \leq 0 \tag{A13}$$

for all $\mathbf{E}_0 \in \mathscr{E}(H_0)$ and all possible inelastic rates $\dot{H}$ that can be realized at $\mathbf{E}_1 \in \partial\mathscr{E}(H_0)$. Similarly, for Il'iushin circuits having $\mathbf{E}_0 = \mathbf{E}_1$ and $\Delta t \ll 1$, it is seen to be necessary to require that

$$(\delta_t^p\mathbf{S})\cdot\dot{\mathbf{E}} \leq 0 \tag{A14}$$

during plastic loading. These necessary conditions are equivalent to those derived by Hill and Rice[12], and subsequently, by Dafalias[5].

APPENDIX B

Let $\mathbf{c}$ represent a standard "metric"-type deformation tensor with $\mathbf{c} = \mathbf{I}$ corresponding to zero deformation and $[\det(\mathbf{c})]^{1/2}$ relating to change of volume. An associated log-strain measure is now defined through the expression

$$2\mathbf{e} \equiv -\ln(\mathbf{c}) = \ln(\mathbf{b}), \qquad \mathbf{b} \equiv \mathbf{c}^{-1}. \tag{B1}$$

This definition clearly implies the relationships

$$\mathbf{c} = \exp(-2\mathbf{e}), \qquad \mathbf{b} = \exp(2\mathbf{e}), \tag{B2}$$

the expansions

$$\mathbf{c} = \sum_{n=0}^{\infty} \frac{1}{n!}(-2\mathbf{e})^n = \mathbf{I} - 2\mathbf{e} + 2\mathbf{e}^2 + \ldots, \qquad \text{for all } \mathbf{e}$$

$$2\mathbf{e} = \sum_{n=1}^{\infty} \frac{1}{n}(\mathbf{I} - \mathbf{c})^n = (\mathbf{I} - \mathbf{c}) + \tfrac{1}{2}(\mathbf{I} - \mathbf{c})^2 + \tfrac{1}{3}(\mathbf{I} - \mathbf{c}) + \ldots, \qquad \text{for all } \mathbf{c}\text{: } \|\mathbf{I} - \mathbf{c}\| < 1 \tag{B3}$$

and the form

$$2\mathbf{e} = -[\ln(c_1)\hat{c}_1 \otimes \hat{c}_1 + \ln(c_2)\hat{c}_2 \otimes \hat{c}_2 + \ln(c_3)\hat{c}_3 \otimes \hat{c}_3] \tag{B4}$$

in terms of the eigenvalues (c_1, c_2, c_3) and corresponding eigenvectors $(\hat{c}_1, \hat{c}_2, \hat{c}_3)$ of $\mathbf{c}$.

One important property of this strain measure is that it contains volume change information in its trace invariant. To see this, consider the respective sets of eigenvalues (c_1, c_2, c_3) and (e_1, e_2, e_3) and note that

$$\begin{aligned}\det(\mathbf{c}) = c_1c_2c_3 &= \exp(-2e_1)\cdot\exp(-2e_2)\cdot\exp(-2e_3)\\ &= \exp[-2(e_1 + e_2 + e_3)]\\ &= \exp[-2\,\mathrm{tr}(\mathbf{e})],\end{aligned}$$

which implies the relationship

$$\mathrm{tr}(\mathbf{e}) = \ln\{[\det(\mathbf{c})]^{-1/2}\}. \tag{B5}$$

Additional considerations pertain to chain rule differentiation under a change of variable from $\mathbf{c}$ to $\mathbf{e}$ in the neighborhood $\| \mathbf{I} - \mathbf{c} \| < 1$. In particular, after noting that

$$\frac{d\mathbf{e}}{d\eta} = -\frac{1}{2}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left[\sum_{\kappa=1}^{n}(\mathbf{I}-\mathbf{c})^{n-\kappa}\frac{d\mathbf{c}}{d\eta}(\mathbf{I}-\mathbf{c})^{\kappa-1}\right]\right\}, \qquad \|\mathbf{I}-\mathbf{c}\|<1,$$

a "double tensor" (linear map from tensors to tensors)

$$\mathbf{G} = \left[\frac{\partial \mathbf{e}}{\partial \mathbf{c}}\right]$$

is defined such that

$$\mathbf{G}\cdot\mathbf{A} \equiv -\frac{1}{2}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left[\sum_{\kappa=1}^{n}(\mathbf{I}-\mathbf{c})^{n-\kappa}\mathbf{A}(\mathbf{I}-\mathbf{c})^{\kappa-1}\right]\right\}, \qquad \|\mathbf{I}-\mathbf{c}\|<1, \tag{B6}$$

and

$$\frac{d\mathbf{e}}{d\eta} = \mathbf{G}\cdot\frac{d\mathbf{c}}{d\eta}. \tag{B7}$$

The case for which $\mathbf{A}$ shares principal directions (and thus commutes) with $\mathbf{c}$ is important since, under these circumstances, (B6) can be shown to reduce to

$$\mathbf{G}\cdot\mathbf{A} \stackrel{*}{=} -\tfrac{1}{2}\mathbf{b}\mathbf{A}. \tag{B8}$$

In view of the identities $\mathbf{A}\cdot\mathbf{BCD} = \mathbf{B}^T\mathbf{A}\mathbf{D}^T\cdot\mathbf{C}$ and $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$, the symmetry of this double tensor is easily established:

$$\begin{aligned}\mathbf{A}\cdot\mathbf{G}\cdot\mathbf{B} &= \mathbf{A}\cdot\left(-\frac{1}{2}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left[\sum_{\kappa=1}^{n}(\mathbf{I}-\mathbf{c})^{n-\kappa}\mathbf{B}(\mathbf{I}-\mathbf{c})^{\kappa-1}\right]\right\}\right)\\ &= -\frac{1}{2}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left[\sum_{\kappa=1}^{n}\mathbf{A}\cdot(\mathbf{I}-\mathbf{c})^{n-\kappa}\mathbf{B}(\mathbf{I}-\mathbf{c})^{\kappa-1}\right]\right\}\\ &= -\frac{1}{2}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left[\sum_{\kappa=1}^{n}(\mathbf{I}-\mathbf{c})^{n-\kappa}\mathbf{A}(\mathbf{I}-\mathbf{c})^{\kappa-1}\cdot\mathbf{B}\right]\right\}\\ &= \mathbf{B}\cdot\left(-\frac{1}{2}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left[\sum_{\kappa=1}^{n}(\mathbf{I}-\mathbf{c})^{n-\kappa}\mathbf{A}(\mathbf{I}-\mathbf{c})^{\kappa-1}\right]\right\}\right)\\ &= \mathbf{B}\cdot\mathbf{G}\cdot\mathbf{A}.\end{aligned} \tag{B9}$$

As a consequence, it follows that if

$$u = f(\mathbf{c}) = h(\mathbf{e}),$$

then

$$\begin{aligned}\frac{du}{d\eta} = \frac{\partial f}{\partial \mathbf{c}}\cdot\frac{d\mathbf{c}}{d\eta} &= \frac{\partial h}{\partial \mathbf{e}}\cdot\frac{d\mathbf{e}}{d\eta}\\ &= \frac{\partial h}{\partial \mathbf{e}}\cdot\mathbf{G}\cdot\frac{d\mathbf{c}}{d\eta}\\ &= \frac{d\mathbf{c}}{d\eta}\cdot\mathbf{G}\cdot\frac{\partial h}{\partial \mathbf{e}}\\ &= \left(\mathbf{G}\cdot\frac{\partial h}{\partial \mathbf{e}}\right)\cdot\frac{d\mathbf{c}}{d\eta},\end{aligned}$$

and thus the "chain rule" result

$$\frac{\partial f}{\partial \mathbf{c}} = \mathbf{G}\cdot\frac{\partial h}{\partial \mathbf{e}}. \tag{B10}$$

The additional result

$$2\left(\frac{\partial u}{\partial \mathbf{c}}\mathbf{c}\right)_{\text{sym}} \equiv \left(\frac{\partial u}{\partial \mathbf{c}}\mathbf{c} + \mathbf{c}\frac{\partial u}{\partial \mathbf{c}}\right) = \mathbf{G}\cdot\left(\frac{\partial u}{\partial \mathbf{e}}\mathbf{c} + \mathbf{c}\frac{\partial u}{\partial \mathbf{e}}\right) \tag{B11}$$

is a simple consequence of (B10) and the defining expression (B6).

As a final exercise, consider two "small strain" expansions. It should be noted that all higher-order terms in the following expressions are of order $\| \mathbf{e} \|^3 \, \| \mathbf{A} \|$.

$$
\begin{aligned}
\mathbf{G}\cdot\mathbf{A} &= -\tfrac{1}{2}\{\mathbf{A} + \tfrac{1}{2}[(\mathbf{I} - \mathbf{c})\mathbf{A} + \mathbf{A}(\mathbf{I} - \mathbf{c})] + \tfrac{1}{3}[(\mathbf{I} - \mathbf{c})^2\mathbf{A} + (\mathbf{I} - \mathbf{c})\mathbf{A}(\mathbf{I} - \mathbf{c}) + \mathbf{A}(\mathbf{I} - \mathbf{c})^2] + \ldots\} \\
&= -\tfrac{1}{2}\{\mathbf{A} + (\mathbf{e} - \mathbf{e}^2)\mathbf{A} + \mathbf{A}(\mathbf{e} - \mathbf{e}^2) + \tfrac{4}{3}(\mathbf{e}^2\mathbf{A} + \mathbf{eAe} + \mathbf{Ae}^2) + \ldots\} \\
\mathbf{G}\cdot\mathbf{A} &= -\tfrac{1}{2}\{\mathbf{A} + (\mathbf{eA} + \mathbf{Ae}) + \tfrac{1}{3}(\mathbf{e}^2\mathbf{A} + 4\mathbf{eAe} + \mathbf{Ae}^2) + \ldots\}
\end{aligned}
\qquad \text{(B12)}
$$

$$
\begin{aligned}
\mathbf{G}\cdot(\mathbf{Ac} + \mathbf{cA}) &= -\tfrac{1}{2}\{(\mathbf{Ac} + \mathbf{cA}) + [\mathbf{e}(\mathbf{Ac} + \mathbf{cA}) + (\mathbf{Ac} + \mathbf{cA})\mathbf{e}] \\
&\quad + \tfrac{1}{3}[\mathbf{e}^2(\mathbf{Ac} + \mathbf{cA}) + 4\mathbf{e}(\mathbf{Ac} + \mathbf{cA})\mathbf{e} + (\mathbf{Ac} + \mathbf{cA})\mathbf{e}^2] + \ldots\} \\
&= -\tfrac{1}{2}\{\mathbf{A}(\mathbf{I} - 2\mathbf{e} + 2\mathbf{e}^2) + (\mathbf{I} - 2\mathbf{e} + 2\mathbf{e}^2)\mathbf{A} + \mathbf{e}[\mathbf{A}(\mathbf{I} - 2\mathbf{e}) + (\mathbf{I} - 2\mathbf{e})\mathbf{A}] \\
&\quad + [\mathbf{A}(\mathbf{I} - 2\mathbf{e}) + (\mathbf{I} - 2\mathbf{e})\mathbf{A}]\mathbf{e} + \tfrac{2}{3}[\mathbf{e}^2\mathbf{A} + 4\mathbf{eAe} + \mathbf{Ae}] + \ldots\} \\
&= -\tfrac{1}{2}\{2\mathbf{A} + \tfrac{2}{3}(\mathbf{e}^2\mathbf{A} - 2\mathbf{eAe} + \mathbf{Ae}^2) + \ldots\} \\
\mathbf{G}\cdot(\mathbf{Ac} + \mathbf{cA}) &= -\{\mathbf{A} + \tfrac{1}{3}(\mathbf{e}^2\mathbf{A} - 2\mathbf{eAe} + \mathbf{Ae}^2) + \ldots\}.
\end{aligned}
\qquad \text{(B13)}
$$